Farey sequence in the appearance of subharmonic Shapiro steps

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The largest Lyapunov exponent has been examined in the dynamical-mode locking phenomena of the ac+dc driven dissipative Frenkel-Kontorova model with deformable substrate potential. Due to deformation, large fractional and higher order subharmonic steps appear in the response function of the system. Computation of the largest Lyapunov exponent as a way to verify their presence led to the observation of the Farey sequence. In the standard regime, the appearance of half-integer and other subharmonic steps between the large harmonic steps, and their relative sizes follow the Farey construction. In the nonstandard regime, however, the half-integer steps are larger than harmonic ones, and Farey construction is only present in the appearance of higher order subharmonic steps. The examination of Lyapunov exponents has also shown that regardless of the substrate potential or deformation, there was no chaos in the system.

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I. INTRODUCTION

In the examination of Shapiro steps, finding the best method or tool to verify their presence has been the matter of many studies in various physical systems. Numerous theoretical and experimental results on Shapiro steps obtained in dissipative systems such as charge- or spin-density wave conductors [1–4], the systems of Josephson junction arrays [5–7], and superconducting nanowires [8,9] have been the main impulse and motivation for our studies of the ac+dc driven overdamped (dissipative) Frenkel-Kontorova (FK) model [10]. It is well known that when these systems are subjected to an external ac driver, their dynamics is characterized by the appearance of Shapiro steps. These steps are due to interference phenomena or dynamical mode locking (synchronization) of the internal frequency with the applied external one. If the locking appears at the integer values of the external frequency, the steps are called harmonic, while for the locking at rational (noninteger) values of frequency they are called subharmonic.

In realistic systems due to presence of noise, impurities, and other environmental effects, detection of Shapiro steps, particularly the subharmonic ones, is usually very difficult. On the other hand, in theoretical works, their observation could also be a problem since their size is often so small that they are invisible on the regular plot of the response function. One of the most sensitive ways to verify the existence of Shapiro steps is the calculation of the *largest Lyapunov exponent* [11]. Always when the system is dynamically mode locked, the largest Lyapunov exponent has negative values [12,13]. Therefore, an examination of the largest Lyapunov exponent for some interval of driven force will precisely reveal the presence of any harmonic or subharmonic mode locking.

Calculation of the largest Lyapunov exponent has been already used as a way to examine the existence of subharmonic Shapiro steps in the standard FK model [12,13]. The standard Frenkel Kontorova (FK) model represents a chain of harmonically interacting particles subjected to a sinusoidal substrate potential [10]. It describes different commensurate or incommensurate structures that, under an external driving force, show very rich dynamical behavior. In the presence of an external ac+dc driving force, the dynamics is characterized by the appearance of the staircase macroscopic response or the Shapiro steps in the response function $\bar{v}(\bar{F})$ of the system [12–14]. Though the standard FK model has been very successful in the studies of some effects related to Shapiro steps, its application is still very restricted. Namely, in the standard FK model, the subharmonic steps either do not exist in the case of commensurate structures with integer values of winding number [15,16] or their size is so small that analysis of their properties is very difficult [12–14]. The absence of subharmonic steps for the commensurate structures with integer value of winding number, and their presence in the case of rational (noninteger) winding number, was confirmed by the calculation of the largest Lyapunov exponent [13]. However, contrary to the standard case, the large subharmonic steps can appear in any commensurate structure of the nonstandard FK model such as the one with the asymmetric deformable substrate potential (ASDP) [17]. This potential belongs to the family of nonlinear periodic deformable potentials, introduced by Remoissent and Peyrard [18] as the way to model many specific physical situations without employing perturbation methods.

In this paper, by using the largest Lyapunov exponent computation technique, we examine the appearance of both harmonic and subharmonic steps in the FK model with assymetric deformable potential (ASDP). In the analysis of the largest Lyapunov exponent, we have observed one interesting property: the Shapiro steps and their relative sizes appear according to the *Farey construction* only in the standard regime when large harmonic steps are dominant in the response function. The paper is organized as follows. The model and methods are introduced in Sec. II, and the results are discussed in Sec. III. Finally, Sec. IV concludes the paper.

II. MODEL AND METHODS

We consider the dissipative (overdamped) dynamics of a series of coupled harmonic oscillators u_l subjected to the

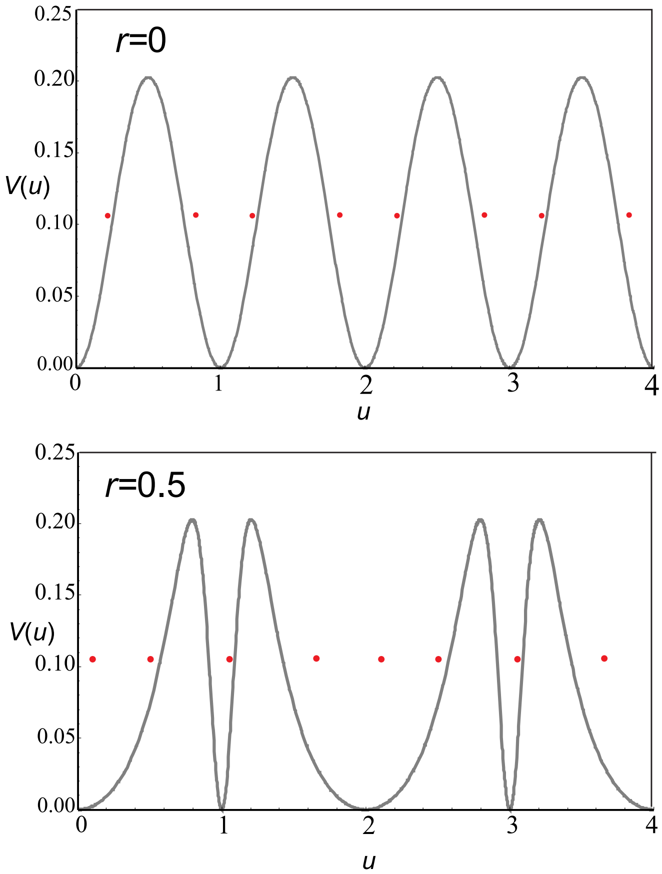

FIG. 1. (Color online) Particles moving in asymmetric deformable potential for $\omega = \frac{1}{2}, K = 4$, and two different values of the shape parameters, $r = 0$ and $r = 0.5$. Particles are represented by red dots.

ASDP:

$$V(u) = \frac{K}{(2\pi)^2} \frac{(1-r^2)^2[1-\cos(2\pi u)]}{[1+r^2+2r\cos(\pi u)]^2}, \tag{1}$$

where K is the pinning strength, and r is the shape or deformation parameter $(-1 < r < 1)$ which can be varied continuously. By changing the shape parameter r the potential can be tuned in a very fine way, from the simple sinusoidal one for $r = 0$ to a deformable one for $0 < r < 1$. In Fig. 1, the commensurate structure $\omega = 1/2$ in the ASDP is presented for two different values of the shape parameter, $r = 0$ and $r = 0.5$ (for more details see [17,19]). The average interparticle distance $\omega = \langle u_{l+1} - u_l \rangle$, or the so-called winding number, is one of the main parameters that describes the FK model. The system exhibits a commensurate phase for rational values of winding number ω, and an incommensurate phase for irrational ones.

In the present paper, the system of coupled harmonic oscillators in the ASDP is driven by the dc+ac forces $F(t) = \bar{F} + F_{\text{ac}}\cos(2\pi\nu_0 t)$. The equations of motion are

$$\dot{u}_l = u_{l+1} + u_{l-1} - 2u_l - \frac{\partial V}{\partial u_l} + F(t), \tag{2}$$

where $l = -\frac{N}{2}, \ldots, \frac{N}{2}$, u_l is the position of the lth particle, and $\bar{F}$ is the dc force, where F_{ac} and $2\pi\nu_0$ are the amplitude and the circular frequency of the ac force, respectively. Since the substrate potential is homogeneous (it does not depend of the particle index l), relabeling of the position of particles does not change the properties of the configuration [10]. When the system is driven by an external ac+dc force, two different frequency scales appear in the system: the frequency of the external periodic force ν_0, and the characteristic frequency of the motion of particles over the ASDP driven by the average force $\bar{F}$. The competition between those frequency scales can result in the appearance of resonance (dynamical mode locking or Shapiro steps).

Solution of the system (2) is called resonant if average velocity $\bar{v}$ satisfies the relation

$$\bar{v} = \frac{i\omega + ja}{m}\nu_0, \tag{3}$$

where the triplet (i, j, m) is made up of integer numbers. The resonant velocity is called harmonic if $m = 1$ and subharmonic if $m \neq 1$. (In the case of $\omega = \frac{1}{q}$ we can use $\bar{v} = \frac{i}{m}\omega\nu_0$ for marking harmonic and subharmonic steps.) Parameter a is the period of the potential $V(u)$, and in the case of no deformation $a = 1$, and with deformation $a = 2$ as can be seen in Fig. 1. For a rational value of $\omega = p/q$ (p and q are co-prime integers) the triplet is not unique (this triplet is unique only for incommensurate structures [10]). In this paper we consider only the commensurate structure $\omega = 1/2$.

The equations of motion (2) have been integrated by using a fourth order Runge-Kutta method with periodic boundary conditions. The time step used in simulations was 0.01τ, where $\tau = \frac{1}{\nu_0}$. The force is varied adiabatically with the step 10^{-5}.

We focus on calculating the largest Lyapunov exponent λ [13]. It is well known that the Lyapunov exponent gives a quantitative measure of the presence of chaos in dynamical systems [11]; however, it also proves to be extremely sensitive to the existence of both harmonic and subharmonic Shapiro steps. When the system is dynamically mode locked, i.e., on the step, the trajectories of particles are periodic in time which is reflected by the negative value of the largest Lyapunov exponent. Outside the steps, where there is no onset of dynamical mode locking, the trajectories are quasiperiodic, which is confirmed by the zero of the Lyapunov exponent [11,13]. We choose an appropriate perturbed point u_l' in our computations according to

$$u_l'(t_{ss}) = u_l(t_{ss}) \pm \sqrt{\frac{d_0^2}{N}}, \tag{4}$$

where t_{ss} is time when the steady state has been achieved in our system, and d_0 is the parameter that expresses the change in the initial positions of particles of the model. In order to make sure that projecting is always done onto the subspace dominated by the largest Lyapunov exponent, the sign in front of the square term in Eq. (4) is randomly selected where the plus and minus sign appear with the same probability. We sample and readjust following Sprott [20] every 25 or so time steps. In our calculations, we used $t_{ss} = 300\tau$ and $d_0 = 10^{-7}$. For convenience, in further text, the largest Lyapunov exponent is denoted just as the Lyapunov exponent.

III. RESULTS

In the present paper, the Lyapunov exponent is examined for different deformations of the substrate potential. In Fig. 2, the Lyapunov exponent as a function of the driving force for three different values of deformation parameter is presented. The insets show the corresponding response functions $\bar{v}(\bar{F})$ (the average velocity as a function of average driving force). As one can see, the computed Lyapunov exponents are always $\lambda \leqslant 0$, which implies that with the change of deformation r we have not introduced chaos in our system (the presence of chaos would result in positive values of Lyapunov exponent). The domain of $\bar{F}$ in Fig. 2, for which we calculated the exponent, differs with r due to the fact that for different values of r the same steps appear in different regions of $\bar{F}$ (see [17,19]). In the standard case in Fig. 2(a), we can see the large minima which correspond to harmonic steps and for which size changes monotonically. As deformation increases in Figs. 2(b) and 2(c), the minima which correspond to the large half-integer and higher order subharmonic steps appear where the changing of their size is not monotonic any more.

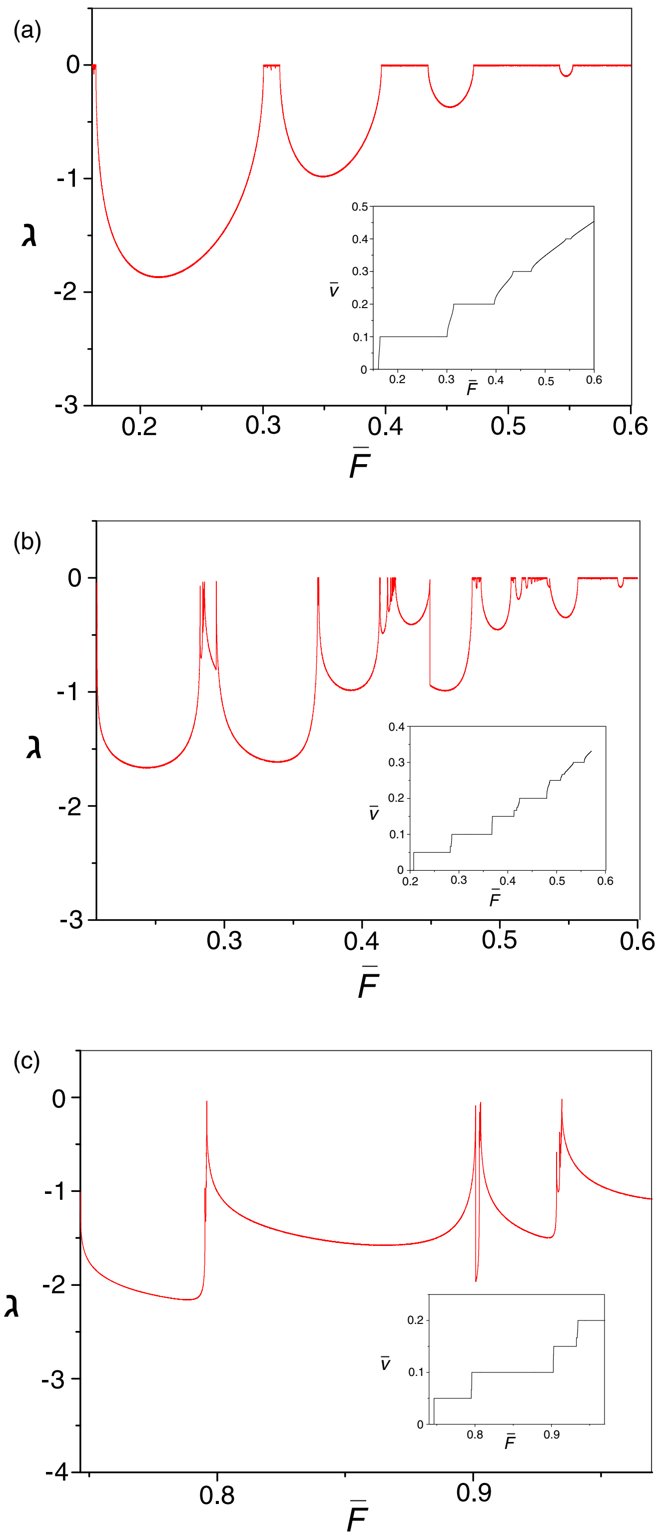

FIG. 2. (Color online) The Lyapunov exponent as a function of the average force for commensurate structure $\omega = 1/2$, $K = 4$, $F_{ac} = 0.2$, $\nu_0 = 0.2$, and three different values of shape parameter (a) $r = 0$, (b) $r = 0.28$, and (c) $r = 0.6$. The insets show the corresponding response functions $\bar{v}(\bar{F})$.

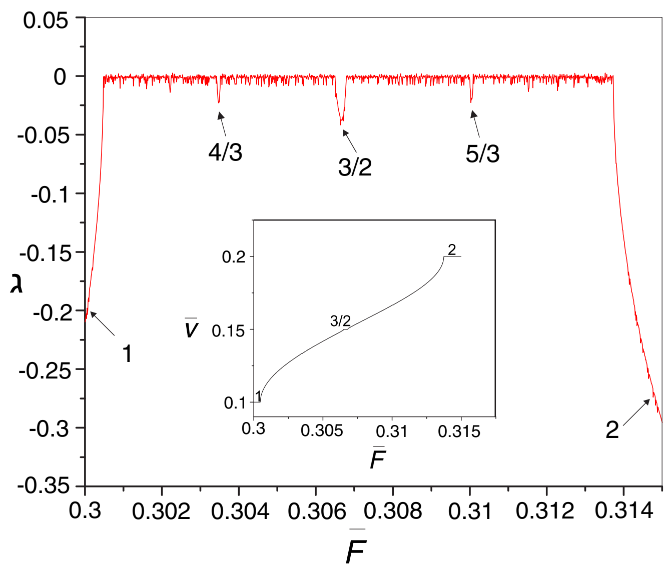

FIG. 3. (Color online) Lyapunov exponent as a function of average driving force for $r = 0$ (the rest of the parameters are as in Fig. 2). The inset shows the response function $\bar{v}(\bar{F})$ drawn for the same interval of force. This result is obtained in [13].

Using Eq. (3), the Shapiro steps could be now identified. It is well known that in the standard FK model ($r = 0$) with integer value of winding number, there would be no subharmonic mode locking [10] and, consequently, no steps between harmonic ones on the plot of response function $\bar{v}(\bar{F})$. On the other hand, when $\omega = 1/2$, only the half-integer step $3/2$ which appears between the first and the second harmonic could be visible [13]. However, computation of the Lyapunov exponent between the first and second harmonic steps reveals other subharmonic steps as can be seen in Fig. 3. The areas under the minima correspond to the size of the steps; i.e., for a larger step, the area under the minimum will be larger. If we examine the subharmonic steps in Fig. 3, we could see that the first largest fractional step between step 1 and step 2 is step $3/2$. Then, the largest step between steps 1 and $3/2$ would be step $4/3$ while the largest one between steps $3/2$ and 2 is $5/3$. Therefore, according to the appearance of fractional steps between the first $1/1$ and the second $2/1$ harmonics we may

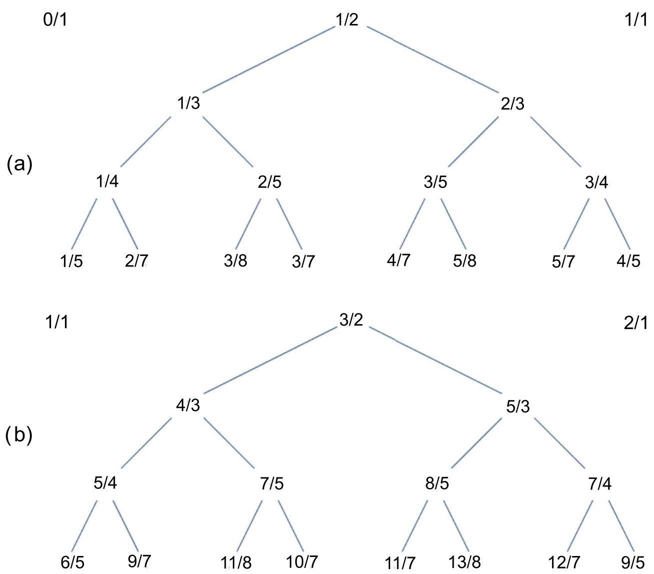

FIG. 4. (Color online) Section of the Farey construction (represented as a rooted binary tree graph) (a) from 0 to 1 and (b) from 1 to 2 according to Eq. (7).

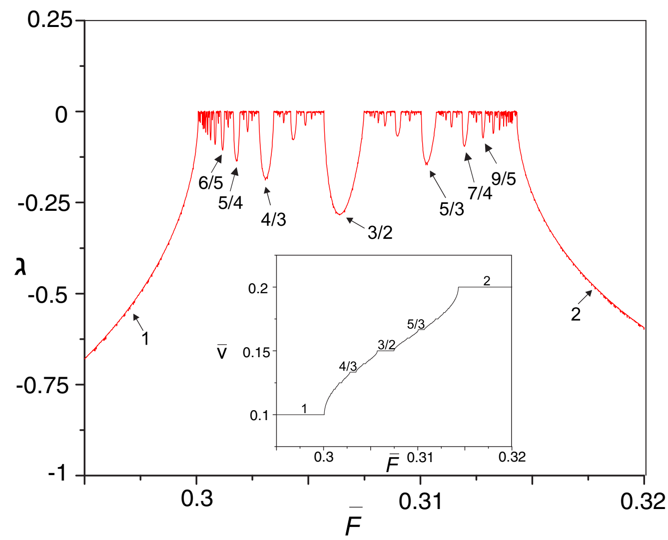

FIG. 5. (Color online) Lyapunov exponent as a function of average driving force for $r = 0.01$ (the rest of the parameters are as in Fig. 2). The inset shows the response function $\bar{v}(\bar{F})$ drawn for the same interval of force.

write the following sequence:

$$\frac{1}{1},\frac{4}{3},\frac{3}{2},\frac{5}{3},\frac{2}{1}. \tag{5}$$

This sequence of numbers represents the *Farey sequence* well known in number theory [21,22].

The Farey sequence $\mathcal{F}_{\mathcal{N}}$ of order $\mathcal{N}$ is an ascending sequence of irreducible fractions between 0 and 1, whose denominators are less than or equal to $\mathcal{N}$ [21,22]. The first few would be

$$\begin{aligned}
\mathcal{F}_1 &= \left\{\tfrac{0}{1},\tfrac{1}{1}\right\},\\
\mathcal{F}_2 &= \left\{\tfrac{0}{1},\tfrac{1}{2},\tfrac{1}{1}\right\},\\
\mathcal{F}_3 &= \left\{\tfrac{0}{1},\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3},\tfrac{1}{1}\right\},\\
\mathcal{F}_4 &= \left\{\tfrac{0}{1},\tfrac{1}{4},\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3},\tfrac{3}{4},\tfrac{1}{1}\right\},\\
\mathcal{F}_5 &= \left\{\tfrac{0}{1},\tfrac{1}{5},\tfrac{1}{4},\tfrac{1}{3},\tfrac{2}{5},\tfrac{1}{2},\tfrac{3}{5},\tfrac{2}{3},\tfrac{3}{4},\tfrac{4}{5},\tfrac{1}{1}\right\}.
\end{aligned} \tag{6}$$

Therefore, if we have two rational fractions in Farey sequence $\frac{p}{q}$ (where p, q are co-prime integers) and $\frac{p'}{q'}$ (where p',q' are co-prime integers), the rational fraction which lies between them and has the smallest denominator is

$$\frac{p''}{q''} = \frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}, \tag{7}$$

where p'', q'' are co-prime integers. This statement is trivially extended to the case of the interval between 1 and 2, and further on (Theorems 28 and 29 in [22]). The largest step between $\frac{p}{q}$ and $\frac{p'}{q'}$, if it exists, will be step $\frac{p}{q} \oplus \frac{p'}{q'}$. The Farey sequence could be easily understood from the diagram in Fig. 4.

For example, in the case of the FK model with the integer value of winding number, there is no subharmonic mode locking, which implies there is only one for the Farey sequence of order. However, if the winding number is a rational noninteger such as the case $\omega = 1/2$ in Fig. 3, one can see that the largest step between harmonic steps 1 and 2 is the half-integer step 3/2. From set theory [22] we know that between any two rational fractions lie countable many, $\aleph_0$ rational fractions and, therefore, countable many possible Shapiro steps between any two harmonic steps in our model.

If the potential gets deformed, the large half-integer step and higher order subharmonic steps appear [17,19]. Contrary to the case $r = 0$ in Fig. 3, now for $r = 0.01$ in Fig. 5, the large 4/3 and 5/3 steps are clearly visible. The higher order subharmonic steps, such as 4/3 and 5/3 (to the left and to the right), are appearing in a symmetric manner with respect to step 3/2.

With the further increase of deformation r, the step widths increase faster on the right side from half-integer step 3/2 than on the left one as can be seen in Fig. 6.

It was shown in our previous work [19] that the sizes of half-integer and subharmonic Shapiro steps increase with deformation and, after reaching their maxima for some value of r, decrease to zero. If we calculate the Lyapunov exponent for $r = 0.28$, which is the value of r for which a half-integer step reaches its maximum, we obtain the results presented in Fig. 7. At this value of deformation some higher subharmonic steps already start to disappear.

At large deformation of the potential, the size of half-integer steps decreases, and the most of higher order subharmonic steps have completely vanished [19]. This is confirmed by the results in Fig. 8, where the Lyapunov exponent for $r = 0.5$ has been calculated. A disappearance of steps is also clearly visible in Fig. 2.

Deformation of the potential obviously strongly affects the steps as we can see in Figs. 6–8. It appears that, with the increase of the deformation r, the right side of the Farey construction is heavily favored over the left one. In particular, we observe that at each level of our binary tree graph (Farey

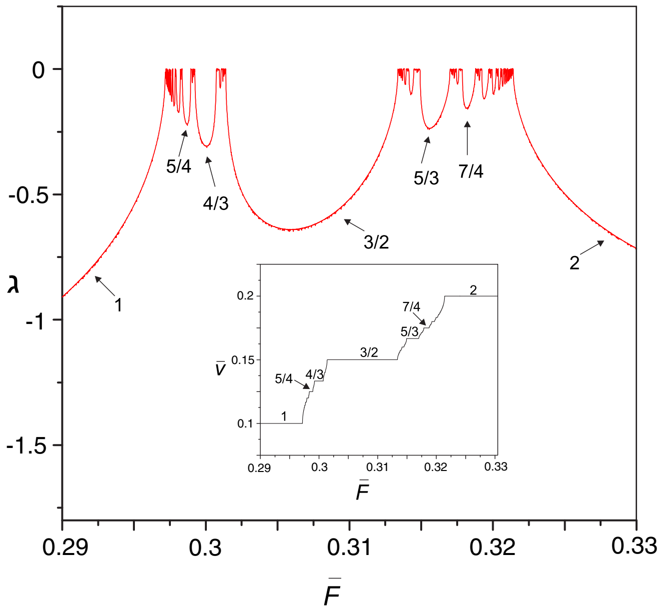

FIG. 6. (Color online) Lyapunov exponent as a function of average driving force for $r = 0.05$ (the rest of the parameters are as in Fig. 2). The inset shows the response function $\bar{v}(\bar{F})$ drawn for the same interval of force.

construction in Fig. 4) the child node (or step) that takes preference is the one on the right. This means that with the increase of the deformation the steps that are present and become increasingly dominant are 3/2, 5/3, 7/4, and 9/5.

We have analyzed also systems with other types of deformable potentials [23], such as variable, double barrier, and double well potential, and we have been always able to observe the appearance of steps in accordance with the Farey construction [11]. Therefore, for two steps $\frac{p}{q}$ and $\frac{p'}{q'}$, the next

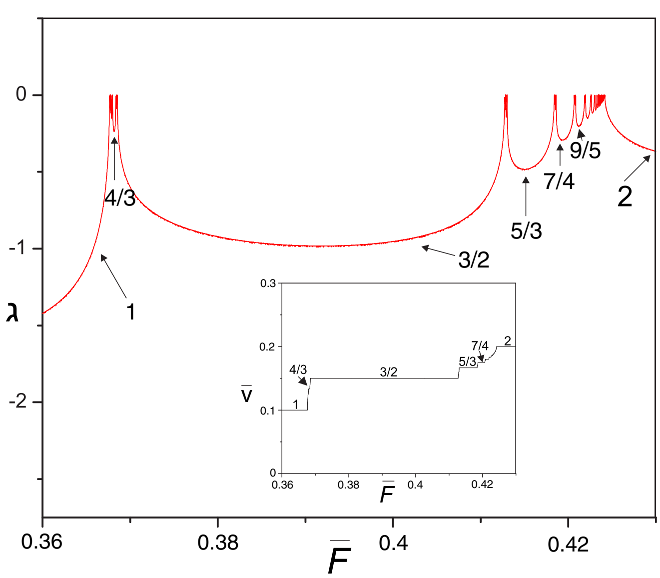

FIG. 7. (Color online) Lyapunov exponent as a function of average driving force for $r = 0.28$ (the rest of the parameters are as in Fig. 2). The inset shows the response function $\bar{v}(\bar{F})$ drawn for the same interval of force.

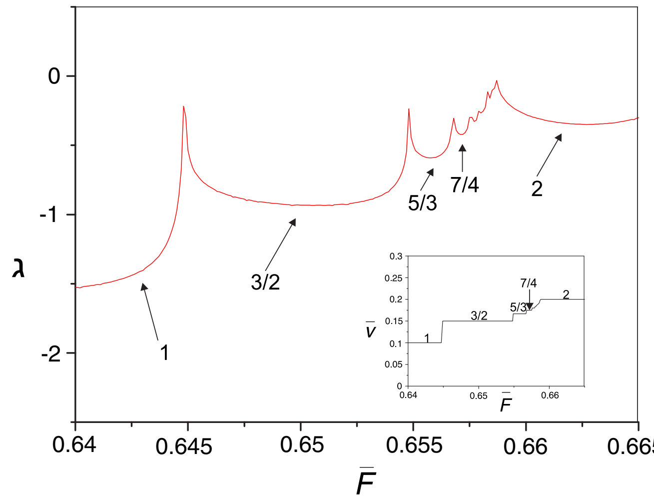

FIG. 8. (Color online) Lyapunov exponent as a function of average driving force for $r = 0.5$ (the rest of the parameters are as in Fig. 2). The inset shows the response function $\bar{v}(\bar{F})$ drawn for the same interval of force.

largest step between them will be $\frac{p+p'}{q+q'}$, where the denominator determines the size of steps in terms that the size of steps decreases as the denominator increases. It is important to note that Farey construction tells us the order and the relative sizes of steps but it does not tell us the actual step width or why they appear [11].

It is well known that the sizes of harmonic and half-integer steps are correlated, whereby the larger the size of harmonic steps the smaller the size of half-integer steps and vice versa [19,24,25]. In some cases, depending on the system parameters, the size of half-integer steps could be even larger than the size of harmonic ones [19]. The size of half-integer and other subharmonic steps strongly affects the behavior of the system, and, according to previous works [19,25], the three different types of system behavior have been classified: the standard behavior for small half-integer steps, the behavior for intermediate half-integer steps, and the behavior in the presence of large half-integer steps.

If we have two harmonic steps, according to the Farey sequence the next largest step which appears between them is the half-integer step, but this is not the case for nonstandard behavior [19,25], since half-integer steps are larger than harmonic ones. In such a case, could we still have the presence of a Farey sequence? In Fig. 9, the response function in the case of large half-integer steps is presented. It is obvious that, in the nonstandard case, the relative sizes of harmonic and half-integer steps do not follow the Farey construction, and in going from harmonic to half-integer steps, the size of the step does not decrease as the denominator increases; on the contrary, the half-integer steps 1/2 and 3/2 are larger than the harmonic ones 1/1 and 2/1. However, the higher order subharmonic steps between half-integer and harmonic steps still appear according to Farey construction and their sizes decrease as the denominator increases.

Calculation of the Lyapunov exponent gives a possibility not only to detect all resonances in the response function, but also to detect the presence of chaos. In all our simulations

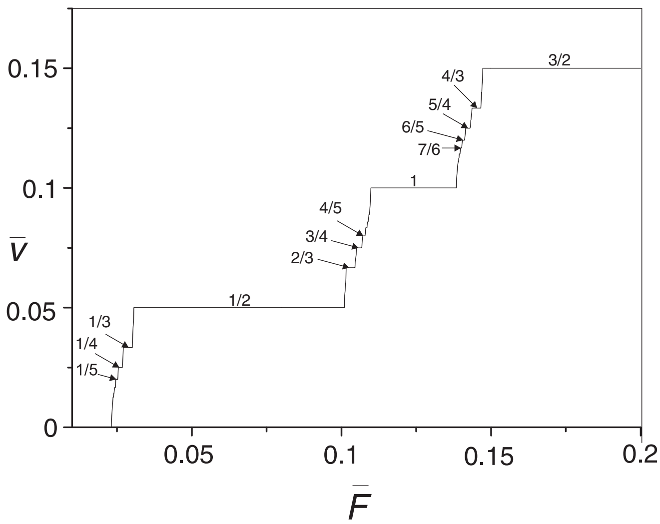

FIG. 9. The average velocity as a function of average driving force for $F_{ac} = 0.55$ and $r = 0.2$ (the rest of the parameters are as in Fig. 2). The numbers mark half-integer and subharmonic steps.

performed on the ac+dc driven overdamped FK model we did not observe any chaos. The presence of deformable substrate potentials and different level of deformations did not introduce chaotic behavior into the system. Contrary to our case, chaos has been observed in the spatiotemporal dynamics of moving kinks in the damped dc driven FK model where the resonances appear due to competition between the moving kinks and their radiated phase modes [26]. Also, structured chaos has been observed in a Josephson junction system where chaotic regions appear between the subharmonic Shapiro steps at certain values of system parameters [27].

IV. CONCLUSION

In this paper, we have presented detailed analysis of the Shapiro steps in the ac+dc driven dissipative FK model by using the Lyapunov computation technique. The obtained results show the presence of a Farey sequence in the appearance of subharmonic steps. The steps and their relative sizes follow exactly the Farey construction only in the standard regime when harmonic steps are the largest one. However, in the nonstandard regime, the half-integer steps are larger than the harmonic ones, and the Farey sequence appears only in the order and relative sizes of higher order subharmonic Shapiro steps. Lyapunov exponent analysis is certainly one of the best ways to get an accurate answer about the presence of chaos in the system. Computations of the Lyapunov exponent have been performed for different system parameters, and, regardless of the deformation, no chaos has been observed in the behavior of the system. The absence of chaos in the presence of deformable potentials certainly requires further investigation. This problem and the possibility of chaotic behavior in other situations such as in the presence of noise will be the subject of future examinations.

The presented results could be important for studies of Shapiro steps in all systems that are closely related to the dissipative dynamics of the FK model. In experimental and theoretical works performed in charge-density wave systems and the systems of Josephson junction arrays, measuring of differential resistance is usually used to detect subharmonic steps. If we look, for example, at the results obtained in sliding charge-density wave systems [2], the systems of Josephson junction arrays [6,7], or superconduction nanowires [8], we can observe the presence of Farey construction in the appearance of Shapiro steps. Our analysis shows that Farey construction can not be always generally applied when it comes to relative sizes of the observed steps. Since the appearance and origin of the subharmonic Shapiro steps are still a matter of debate, we hope that these results could bring insight into understanding of these physical phenomena.

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