Harvesting stabilizer entropy and non-locality from a quantum field

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The harvesting of quantum resources from the vacuum state of a quantum field is a central topic in relativistic quantum information. While several proposals for the harvesting of entanglement from the quantum vacuum exist, less attention has been paid to other quantum resources, such as non-stabilizerness, commonly dubbed *magic* and quantified by the Stabilizer Rényi Entropy (SRE) [1]. In this work, we show how to harvest SRE from the vacuum state of a massless field, using accelerated Unruh-DeWitt detectors in Minkowski spacetime. In particular, one can harvest a particular non-local form of SRE that cannot be erased by local unitary operations. This non-local SRE is a fundamental quantity to study the interplay between entanglement and non-stabilizer resources. We conclude our work with an analysis of the CHSH inequalities: one cannot extract a violation from the quantum field even when one can harvest the necessary resources for it.

Introduction.— The harvesting of quantum resources from the vacuum state of a quantum field [2–5] plays a prominent role in the interplay of quantum information theory with quantum field theory, due to its applications in quantum information processing [6, 7], quantum gravity [8–13] and newly discovered phenomena such as energy teleportation [14, 15].

Harvesting consists in extracting a quantum resource originally located in a quantum system, e.g., a quantum field, where it cannot be exploited, and moving it into another quantum system, e.g., a detector, from which the harvested resource can then be used to perform useful tasks. The most prominent and known example of these protocols is entanglement harvesting: the vacuum state of a free field is entangled when seen by an accelerating observer [16, 17] and through an entanglement harvesting protocol it is possible to move the entanglement from the field to a pair of detectors. This kind of protocol has been analyzed in many different settings, as one can change the kind of detector used for the harvesting [18, 19] or consider causally disconnected [20] or accelerated detectors [21, 22].

Beyond entanglement, more general quantum resources can be harvested from quantum fields [23–25]. In this Letter, we propose a novel protocol aiming at harvesting non-stabilizerness, colloquially dubbed *magic*. Non-stabilizerness is the fundamental resource necessary to achieve quantum advantage in computational tasks, and it has thus received great attention for its role in quantum information processing and quantum complexity. Indeed, from a complexity perspective, even maximally entangled states can be efficiently simulated using classical resources [26–28], as long as they do not possess resources beyond stabilizer ones.

As non-stabilizerness is a fundamental resource to achieve quantum advantage, many measures for its quantification have been proposed [29–32], such as the Wigner negativity [33], the stabilizer rank [34] and the Stabilizer Rényi Entropy [1]. In particular, the SRE is the unique computable magic monotone for pure states [35] and it is computed from the expectation values of a quantum state over the Pauli strings. SRE can be computed in an efficient way by perfect sampling [36, 37] and can be extended both to qudits [38] and mixed states. This resource is also experimentally measurable [39].

Non-stabilizerness is a central quantity in scenarios that go beyond quantum information processing. It has been proved that it plays a role in Conformal Field Theories [40] and in processes of state decoding from a black hole [41, 42]. Non-stabilizerness has been recently shown to be a necessary ingredient in the context of simulations of quantum gravity models [43, 44], nuclear physics [45, 46] and dense neutrinos systems [47].

If one restricts to Pauli measurements, non-stabilizer resources are also necessary to violate the Clauser-Horne-Shimony-Holt (CHSH) inequalities [48]. This restriction corresponds to the restriction to local measurements in establishing the importance of entanglement for such violations.

In this paper, we study the harvesting of both SRE and entanglement from the vacuum of a scalar field in an accelerated reference frame. We also ask whether non-locality in the form of CHSH violation can be harvested in this scenario. The

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answer is that one can harvest SRE from a quantum field but there is no way that one, starting with a resource-free state - that is, a state without SE and EE - can violate CHSH inequalities by resource harvesting. Moreover, even if one starts with an initial state containing one of the two resources, there is no way of extracting a violation of CHSH inequalities. In [49], the authors study the behavior of the Bell operator in a the interaction with quantum fields, but not with the goal of harvesting. Starting with a state that violates CHSH inequalities, they find that the violation decreases for the interaction with a quantum field.

In [50] a magic harvesting protocol has been proposed using a three-level Unruh-DeWitt detector and a quantifier known as mana [33]. Our work differs from [50] and goes beyond it in at least two aspects. First, as clarified by the authors, in [50] the appearance of non-stabilizerness in the detector's final state is due *solely* to the interaction of the detector with the quantum field, so there is no actual harvesting of the resource. In this paper, we harvest the non-stabilizer resource from the field in two ways: in a *weak sense*, because we show that the protocol would not be able to induce non-stabilizer resources if the quantum field did not possess them and, in a *strong sense*, that the extraction of a particularly strong form of SRE, the so-called *non-local* SRE[43] is extracted through a protocol that is a free operation for the associated resource theory. Second, as we use SRE in place of mana to quantify non-stabilizerness, we are able to consider arbitrary states of the detectors. This allows us to consider a different physical setting, e.g. two causally disconnected observers moving with two accelerating reference frames with parallel, antiparallel and perpendicular mutual acceleration.

Harvesting protocol.— We consider two detectors A and B moving in an accelerated reference frame with respect to each other. These are modeled as two level systems with Hamiltonian:

$$H_{AB} = H_A + H_B = \frac{\Omega}{2} Z_A + \frac{\Omega}{2} Z_B, \tag{1}$$

where Z_i with $i = A, B$ is the diagonal Pauli operator acting on the Hilbert spaces of system A and B respectively. We consider a massless scalar field ϕ in Minkowski space-time described by the free Hamiltonian:

$$H_\phi = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \phi^\dagger_{\boldsymbol{p}} \phi_{\boldsymbol{p}}. \tag{2}$$

where $\boldsymbol{p}$ is the momentum of the field.

The two detectors interact with the quantum field via the interaction Hamiltonian (in the interaction picture):

$$H^{(i)}_{\text{int}} = \lambda \epsilon_i(\tau) \mu_i(\tau) \otimes \phi(x_i(\tau)), \tag{3}$$

where $i = A, B$ and τ is the proper time. Here λ is a parameter gauging the strength of the interaction and $\epsilon_i(\tau)$ is a switching function describing the turning on and off of the interaction. The operator $\phi(x_i(\tau))$ is the field operator evaluated along the space-time trajectory of detector i. Finally, the operator $\mu_i(\tau)$ acts on the detector as:

$$\mu_i(\tau) = e^{-i\Omega\tau} |1\rangle\langle 0| + e^{+i\Omega\tau} |0\rangle\langle 1| \,. \tag{4}$$

Notice that in contrast with entanglement harvesting protocols, in principle one does not need two detectors to harvest non stabilizer resources, since this resource can be highly non-trivial even for a single qubit. However, we chose to work with two detectors as this allows us to study also the interplay between entanglement and non-stabilizerness. Furthermore, the presence of two different detectors allows us to study the CHSH inequality in the quantum relativistic setting.

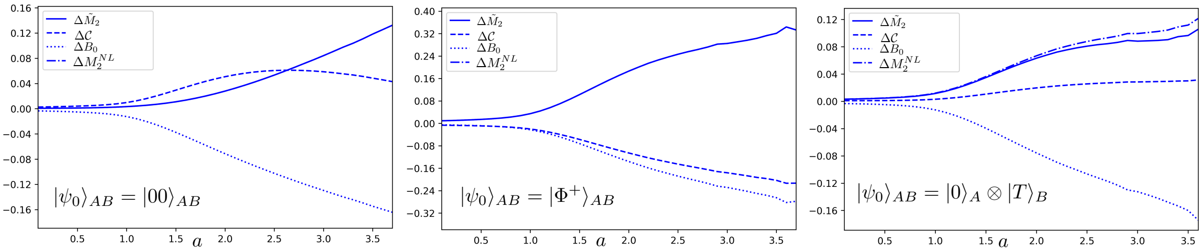
The state of the joint system evolves in the interaction picture according to the Hamiltonian in Eq. (3), so that we can write the unitary operator U describing the dynamics of the system as:

$$U = \mathcal{T} \exp\left[-i\left(\int d\tau_A H^{(A)}_{\text{int}} + \int d\tau_B H^{(B)}_{\text{int}}\right)\right]. \tag{5}$$

One can then compute the final state of the joint system as $|\psi_f\rangle = U|\psi_0\rangle$. Since we are interested in the resources of the detector state, we can trace out the field degrees of freedom and obtain the final (mixed) state of the detectors $\rho_{AB} = \text{Tr}_\phi |\psi_{\text{f}}\rangle\langle\psi_{\text{f}}|$. For instance, in the case the detectors are initialized in the state $|00\rangle$ one gets the following expression for ρ_{AB}:

$$\begin{pmatrix} 1 - |E_A|^2 - |E_B|^2 - X & 0 & 0 & M \\ 0 & |E_B|^2 & \langle E_A|E_B\rangle & 0 \\ 0 & \langle E_B|E_A\rangle & |E_A|^2 & 0 \\ M^* & 0 & 0 & X \end{pmatrix}$$

where $|E_i|^2$ is the probability of having the i-th detector excited after the interaction, $\langle E_B|E_A\rangle$ is the overlap between the excited states of the two detectors and M is the probability that the two detectors exchange a virtual particle. Once the parameters of the interaction are fixed, all terms only depend on the trajectories of the detectors, i.e. on the relative acceleration a. We show the full expressions of these terms together with their physical meaning in [51]. It is important to note that this matrix is a bit different from the one usually appearing in the entanglement harvesting literature.



(a) (b) (c)

Figure 1: Variations of SRE $\Delta\tilde{M}_2(\rho_{AB})$ (solid line), concurrence $\Delta\mathcal{C}(\rho_{AB})$ (dashed line), non-local SRE $\Delta M_2^{NL}(\phi)$ (dashed-dotted line) and value of Bell operator ΔB_0 (dotted line) as a function of the (parallel) acceleration $a_\parallel$. Panels (a) Initial state $|00\rangle$, (b) initial state $|\Phi^+\rangle$, (c) initial state $|0\rangle_A \otimes |T\rangle_B$. In panel (a) and (b) the curves for $\Delta\tilde{M}_2(\rho_{AB})$ and $\Delta M_2^{NL}(\phi)$ overlap perfectly as all the harvested SRE is non-local.

In fact, all the terms appearing in the density matrix are of second order in λ, except for the term $X \equiv |E_A|^2|E_B|^2 + |\langle E_A|E_B\rangle|^2 + |M|^2$: this is of fourth order, and it can be usually neglected in entanglement harvesting protocols, as it plays no role for the evaluation of the concurrence. However, as SRE needs a properly defined density matrix in order to be computed, here we need to retain the term X in order to preserve the positivity of the density matrix. We fix the interaction parameters to $\Omega = 2$, $\sigma = 1$ and $L = 0.5$, where Ω is the detector energy gap, σ is the switching time of the interaction and L is the initial relative distance of the detectors. Then the density operator ρ_{AB} is a function of the acceleration a only.

At this point, three main scenarios can be individuated, depending on the direction of acceleration of the two detectors, i.e., *parallel* $a_\parallel$, *antiparallel* $a_{\nparallel}$ and *perpendicular* $a_\perp$ acceleration. The only difference between these scenarios is in the calculation of the matrix elements, while the overall structure of the reduced density matrix of the detectors only depends on the initial state of the detectors. As we shall see, SRE is always harvested from the field, proving that acceleration is indeed the source responsible for the generation of SRE. We focus here on the parallel scenario, leaving the antiparallel and perpendicular cases in [51].

Harvesting of SRE and entanglement.— As the state ρ_{AB} of the two detectors after the interaction with the field is mixed, some care is needed in evaluating both SRE and entanglement. In the case of two spins (qubit), a good monotone is the concurrence $\mathcal{C}$ [52, 53], defined as:

$$\mathcal{C} := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \quad (6)$$

where λ_i are the eigenvalues in decreasing order of the matrix $R = \sqrt{\rho^{1/2}\tilde{\rho}\rho^{1/2}}$ and $\tilde{\rho} = (Y \otimes Y)\rho^*(Y \otimes Y)$.

We quantify the amount of non-stabilizerness in the detectors final state via the 2-SRE, which for pure states is defined as [1]:

$$M_2(|\psi\rangle) := -\log d \,\mathrm{tr}\big[Q|\psi\rangle\langle\psi|^{\otimes 4}\big] \quad (7)$$

where $Q := d^{-2}\sum_{P\in\mathcal{P}} P^{\otimes 4}$, d is the dimension of the Hilbert space and $\mathcal{P}$ indicates the set of all n qubits Pauli strings, i.e. strings of the form $P = P_1 \otimes P_2 \otimes \cdots \otimes P_n$, with $P_i \in \{X, Y, Z, I\}$. The 2-SRE can be extended to generic mixed states as

$$\tilde{M}_2(\rho) = M_2(\rho) - S_2(\rho), \quad (8)$$

where $S_2(\rho) = -\log \mathrm{tr}\big[\rho^2\big]$ is the 2-Rényi entropy of ρ. Plugging the expression for the final state of the detectors ρ_{AB} into Eq. (8) one can find an expression for the SRE $\tilde{M}_2(\rho_{AB})$ as a function of the acceleration a.

It is important to remark here that for mixed states SRE does not have the usual meaning of distillable non-stabilizer resources. In this sense, it behaves exactly like the von Neumann entropy for entanglement. However, it has the operational meaning of quantifying the hardness of certain quantum information protocols [54–56]. Moreover, the resource free states $\tilde{M}_2(\rho) = 0$ are those that cannot be purified in a stabilizer state[26, 43]. The quantity $\tilde{M}_2$ is in fact a good *proxy* for this resource theory[57].

The *non-local* non-stabilizerness can be defined [43] as $M^{NL}(\phi) := \min_{R=U_A\otimes U_B} M(R(\phi))$ for any non-stabilizer monotone M. Non-local SRE is obtained by using SRE as non-stabilizer measure. M^{NL} represents the non-stabilizer resources that cannot be erased by local unitary operations. It plays an important role in AdS-CFT as they represent the holographic dual of back-reaction. As $M(\phi)^{NL}$ vanishes identically on both product and stabilizer states, it quantifies the non-locality of

SRE and serves as a useful probe to investigate the interplay between entangling and non-stabilizer resources. Notice that a LOCC protocol would not be able to extract non-local SRE M^{NL}. So, to the extent that there is genuine harvesting of entanglement, there is genuine harvesting of M^{NL} in the strong sense.

CHSH inequalities.— Non-stabilizerness is also a necessary ingredient in order to violate the CHSH formulation of Bell's inequality. Starting with the state $\omega = |00\rangle\langle 00|$ and considering a Bell operator $\mathcal{B}_0$ of the form

$$\mathcal{B}_0 = P_A^{(1)}\left(P_B^{(1)} + P_B^{(2)}\right) + P_A^{(2)}\left(P_B^{(1)} - P_B^{(2)}\right) \quad (9)$$

with all the $P_i^{(j)} \in \{X, Y, Z\}$, one easily obtains:

$$b_0 = \mathrm{Tr}\left[\mathcal{B}_0 \mathrm{C}\omega\mathrm{C}^\dagger\right] \leq 2 \quad (10)$$

where C is any unitary operation belonging to the Clifford group. The meaning of Eq. (10) is that entanglement alone is not sufficient to violate the CHSH inequality. Notice that this is not in contrast with the usual setting of Bell's inequality in the CHSH setting, where measurements in any direction are allowed, which in general require non stabilizer resources to be performed. One is then tempted to check whether it is possible to harvest enough SRE and entanglement as to violate the CHSH inequality, since this would imply the possibility of harvesting non-locality.

In the following, we restrict our measurements to be both local and Pauli. In this way, there are no resources that go beyond Clifford or locality. We thus define a Bell operator

$$\mathcal{B}_0 := X_A X_B + X_A Z_B - Z_A X_B + Z_A Z_B \quad (11)$$

whose maximum value over the set of stabilizer states is obtained for the maximally entangled state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $\mathrm{tr}(\mathcal{B}_0\Phi^+) = 2$. We define $b_0 \equiv \mathrm{Tr}[\rho_{AB}(0)\mathcal{B}_0]$ the initial value of the Bell operator.

Results.— As mentioned in the previous section, here we focus only on the parallel acceleration scenario. Results for other accelerations are similar and can be found in [51]. Starting from different initial states of the detectors, we compute the variation of SRE $\Delta\tilde{M}_2(\rho_{AB})$, of the concurrence $\Delta\mathcal{C}(\rho_{AB})$, of the expectation value over the Bell operator in Eq. (11) ΔB_0, and variation $\Delta M_2^{NL}(\phi)$ of the non-local SRE $M_2^{NL}(\phi) := \min_{R=U_A\otimes U_B} \tilde{M}_2(R(\phi))$.

Specifically, we consider three different initial states $|\psi_0\rangle_{AB}$ of the detectors: the resource-free state $|00\rangle_{AB}$, the maximally entangled, SRE-free state $|\Phi^+\rangle_{AB}$ and the separable, SRE-ful state $|0\rangle_A \otimes |T\rangle_B$ with $|T\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$. We summarize the initial values of the quantities of interest for the different initial states in Table I. After interaction with the field for the observer with acceleration a, we compute (see Fig. 1a, 1b, 1c) we the variations $\Delta\tilde{M}_2(\rho_{AB}), \Delta\mathcal{C}(\rho_{AB}), \Delta B_0, \Delta M_2^{NL}(\phi)$ as a function of the acceleration a for the three choices of initial states, respectively. In all cases, one harvests both entanglement $\mathcal{C}$ and SRE $\tilde{M}_2$.

$\rho_{AB}(0)$	$\tilde{M}_2$	$\mathcal{C}$	b_0	M_2^{NL}
$\|00\rangle$	0	0	1	0
$\|\Phi^+\rangle$	0	1	2	0
$\|0\rangle \otimes \|T\rangle$	$\simeq 0.415$	0	$-\frac{\sqrt{2}}{2}$	0

Table I: Initial values of SRE $\tilde{M}_2$, concurrence $\mathcal{C}$, Bell operator b_0 and non-local SRE M_2^{NL} for different choices of the initial state of the detectors.

We see that the harvesting of resources is non-monotone with the acceleration $a_\parallel$. Remarkably, if there is no SRE in the initial state, *all* the harvested SRE is non-local, so this is genuine harvesting in the strong sense. Even if there is some local SRE in the initial state, most of the harvested SRE is still non-local, see fig.1c. In this latter case, even the local part of the harvested SRE, would not be produced if acceleration is zero, and the field would be in a stabilizer state. At finite acceleration, the SRE in the field enables the production of SRE in the detectors: this is harvesting in the weak sense[58].

Importantly, in all cases the CHSH cannot be violated. We remark here that similar results are obtained, especially regarding CHSH, for any choice of Pauli measurement in the definition of the Bell operator $\mathcal{B}_0$, Eq. (11). If one starts with resource-free states, there is no enough harvesting of the resources to guarantee a CHSH violation, if one saturates one of the resources, e.g., entanglement or SRE (on one of the detectors), the interaction with the field loses the resource as the other one is harvested. In a way or another, the form of non-locality displayed by CHSH violations cannot be induced by interacting with the field.

To conclude, one can harvest substantial amounts of SRE and entanglement, and even non-local SRE. However, it is not possible to harvest non-locality in the form of CHSH violation from the quantum field.

Conclusions and outlook.— In this letter, we have shown a protocol for SRE harvesting in accelerated reference frames. The main result of the paper is that SRE can be harvested from the vacuum state of a relativistic quantum field and stored into the state of a detector interacting with the field. Moreover, SRE and entanglement can be harvested at the same time, including in the particularly strong form of non-local SRE.

We have also shown that, at least with this protocol, it is not possible to harvest non-locality from the field, since there is either not enough entanglement or enough SRE. Even starting with a maximally entangled state, one can harvest SRE in non-local form, but a trade-off of resources happens so that the CHSH inequalities can never be violated. In perspective, we are going to address the question whether there is at play a principle of fundamental nature, a no-go theorem for the harvesting of true non-locality for a quantum field. Similarly, one wonders how steering and the Reeh-Schlieder theorem can be used to transfer state preparation in the field to state preparation of detectors[59].

It would be interesting to study how the harvested SRE depends on the dimensionality of the detectors. Then one could also consider more realistic scenarios, for instance, studying the case of finite detectors rather than point like, or with different switching functions. Another interesting possibility is to consider massive scalar field moving along a trajectory in curved space-time instead of the simple setting of Minkowski background.

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Appendix A: Computation of the reduced density matrix of the detectors

In this section, we show how to derive the reduced density matrix of the detectors when they are initialized in the state $|00\rangle$. For different choices of the initial state, the procedure goes exactly the same.

Let us introduce the following notation to shorten significantly the expressions:

$$\epsilon_i(t) = \epsilon(\tau_i(t))\frac{d\tau_i}{dt}, \quad \phi_i(t) = \phi(x_i(t)), \quad \phi_i{}^{\pm} = \int dt \epsilon_i(t) e^{\pm i\Omega\tau_i(t)}\phi_i, \quad \left|E_i^{\pm}\right\rangle = \phi_i^{\pm}\left|0\right\rangle_{\phi}$$

where $i \in \{A, B\}$. Using this notation, the interaction Hamiltonian reads:

$$H_i = \lambda\epsilon_i(t)(e^{i\Omega\tau_i(t)}\left|1\right\rangle\left\langle 0\right| + e^{-i\Omega\tau_i(t)}\left|0\right\rangle\left\langle 1\right|) \otimes \phi_i(t) \tag{A1}$$

The evolution occurs in both proper times τ_A and τ_B, and for small value of the coupling constant $\lambda << 1$, the final state can be expanded as

$$\left|\psi_f\right\rangle = U\left|\psi_i\right\rangle = \sum_n \lambda^n \left|\psi_f^{(n)}\right\rangle \tag{A2}$$

To show this explicitly, we write U in terms of the coordinate time t, with respect to which the vacuum state of the field is defined.

We expand U in Dyson series up to the second order:

$$\begin{aligned} U &= \mathcal{T}\exp\left[-i\int dt\Big(\frac{d\tau_A}{dt}H_A(\tau_A(t)) + \frac{d\tau_B}{dt}H_B(\tau_B(t))\Big)\right] \\ &= \mathbb{1} - i\int dt\Big(\frac{d\tau_A}{dt}H_A(\tau_A(t)) + \frac{d\tau_B}{dt}H_B(\tau_B(t))\Big) \\ &- \frac{1}{2}\int dt\int dt'\mathcal{T}\Big[\frac{d\tau_A}{dt}H_A(\tau_A(t))\frac{d\tau_A}{dt'}H_A(\tau_A(t')) + \frac{d\tau_B}{dt}H_B(\tau_B(t))\frac{d\tau_B}{dt'}H_B(\tau_B(t')) \\ &+ \frac{d\tau_A}{dt}H_A(\tau_A(t))\frac{d\tau_B}{dt'}H_B(\tau_B(t')) + \frac{d\tau_B}{dt}H_B(\tau_B(t))\frac{d\tau_A}{dt'}H_A(\tau_A(t'))\Big] \end{aligned} \tag{A3}$$

Before showing the perturbative terms of the final state $|\psi_f\rangle$, let us introduce the additional notation that will be useful in the following:

$$\phi_i^{\pm}(t) = \int_{-\infty}^{t} dt' \epsilon_i(t') e^{\pm i\Omega\tau_i(t')} \phi_i(t'), \qquad \left|E_i^{\pm}(t)\right\rangle = \phi_i^{\pm}(t) \left|0\right\rangle_\phi$$

We can then compute the various terms of the series expansion.

- $O(\lambda^0)$

$$\left|\psi_f^{(0)}\right\rangle \;=\; \mathbb{1}\left|\psi_i\right\rangle = \left|00\right\rangle\left|0\right\rangle_\phi \tag{A4}$$

- $O(\lambda^1)$

$$\begin{aligned}\left|\psi_f^{(1)}\right\rangle &= -i\int dt\Big(\frac{d\tau_A}{dt}H_A(\tau_A(t)) + \frac{d\tau_B}{dt}H_B(\tau_B(t))\Big)\left|\psi_i\right\rangle \\ &= -i\lambda\int dt\Big[\epsilon_A(t)(e^{i\Omega\tau_A(t)}\left|1\right\rangle\left\langle 0\right| + e^{-i\Omega\tau_A(t)}\left|1\right\rangle\left\langle 0\right|)\otimes\phi_A(t) \\ &+ \epsilon_B(t)(e^{i\Omega\tau_B(t)}\left|0\right\rangle\left\langle 1\right| + e^{-i\Omega\tau_B(t)}\left|1\right\rangle\left\langle 0\right|)\otimes\phi_B\Big]\left|00\right\rangle\left|0\right\rangle_\phi \\ &= -i\lambda\int dt\Big[\epsilon_A(t)e^{i\Omega\tau_A(t)}\left|10\right\rangle\otimes\phi_A(t)\left|0\right\rangle_\phi \\ &+ \epsilon_B(t)e^{i\Omega\tau_B(t)}\left|01\right\rangle\otimes\phi_B(t)\left|0\right\rangle_\phi\Big] \qquad \text{(A5)} \\ &= -i\lambda\left(\left|10\right\rangle\left|E_A^{+}\right\rangle + \left|01\right\rangle\left|E_B^{+}\right\rangle\right) \qquad \text{(A6)}\end{aligned}$$

- $O(\lambda^2)$

$$\begin{aligned}\left|\psi_f^{(2)}\right\rangle &= -\frac{\lambda^2}{2}\int dt\int dt'\mathcal{T}\Bigg[\frac{d\tau_A}{dt}\frac{d\tau_A}{dt'}H_A(t)H_A(t') + \frac{d\tau_A}{dt}\frac{d\tau_B}{dt'}H_A(t)H_B(t') \\ &+ \frac{d\tau_B}{dt}\frac{d\tau_A}{dt'}H_B(t)H_A(t') + \frac{d\tau_B}{dt}\frac{d\tau_B}{dt'}H_B(t)H_B(t')\Bigg]\left|\psi_i\right\rangle \qquad \text{(A7)}\end{aligned}$$

Let us look first at one of these four integrals. For the term proportional to $H_A(t)H_A(t')$ we get:

$$\begin{aligned}&\int dt\int dt'\frac{d\tau_A}{dt}\frac{d\tau_A}{dt'}H_A(t)H_A(t')\left|\psi_i\right\rangle = \int dt\int_{-\infty}^{t} dt'\frac{d\tau_A}{dt}\frac{d\tau_A}{dt'}H_A(t)H_A(t')\left|00\right\rangle\left|0\right\rangle_\phi \\ &= \int dt\int_{-\infty}^{t} dt'\epsilon_A(t)\epsilon_A(t')\Big(e^{i\Omega\tau_A(t)}\left|1\right\rangle\left\langle 0\right|_A + e^{-i\Omega\tau_A(t)}\left|0\right\rangle\left\langle 1\right|_A\Big)\Big(e^{i\Omega\tau_A(t')}\left|1\right\rangle\left\langle 0\right|_A \\ &+ e^{-i\Omega\tau_A(t')}\left|0\right\rangle\left\langle 1\right|_A\Big)\left|00\right\rangle\phi_A(t)\phi_A(t')\left|0\right\rangle_\phi \\ &= \int dt\int_{-\infty}^{t} dt'\epsilon_A(t)\epsilon_A(t')e^{-i\Omega\tau_A(t)}e^{i\Omega\tau_A(t')}\phi_A(t)\phi_A(t')\left|0\right\rangle_\phi\left|00\right\rangle \\ &= \int dt\epsilon_A(t)e^{-i\Omega\tau_A(t)}\phi_A(t)\left|E_A^{+}(t)\right\rangle\left|00\right\rangle \\ &= \left|00\right\rangle\phi_A^{-}\left|E_A^{+}(t)\right\rangle \qquad \text{(A8)}\end{aligned}$$

Notice that the operator ϕ_A^- act on the state $\left|E_A^+(t)\right\rangle$ performing an integration over the time variable, ensuring that the resulting state would not depend on t.

The other combinations of Hamiltonians can be computed in the same way as above. One can show that the final state at the second order in λ is given by:

$$\left|\psi_f^{(2)}\right\rangle = -\Big(\left|00\right\rangle\phi_A^{-}\left|E_A^{+}(t)\right\rangle + \left|11\right\rangle\phi_A^{+}\left|E_B^{+}(t)\right\rangle + \left|11\right\rangle\phi_B^{+}\left|E_A^{+}(t)\right\rangle + \left|00\right\rangle\phi_B^{-}\left|E_B^{+}(t)\right\rangle\Big) \tag{A9}$$

Since we are interested in harvesting resources from the detector state, we have to trace out the degrees of freedom of the field $\rho_{AB} = Tr_\phi \left|\psi_f\right\rangle \left\langle\psi_f\right|$ Taking terms up to the second order in λ, one obtains the following density matrix:

$$\rho_{AB} = \begin{pmatrix} 1-|E_A|^2-|E_B|^2 & 0 & 0 & M \\ 0 & |E_B|^2 & \langle E_A|E_B\rangle & 0 \\ 0 & \langle E_B|E_A\rangle & |E_A|^2 & 0 \\ M^* & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{A10}$$

where $|E_i|^2$ is the probability of having the i-th detector excited after the interaction, $M = \langle E_B^-|E_A^+\rangle - \langle E_B^+|E_A^-\rangle$ is the probability that the two detectors exchange a virtual particle and $\langle E_B|E_A\rangle = \langle E_B^+|E_A^+\rangle$ is the overlap between the excited states of the two detectors.

This reduced density matrix is perfectly fine if one is interested in computing only the concurrence of the state, however the same is not true when computing the SRE. In fact, in order to correctly compute the SRE one needs a fully legitimate, i.e. positive and trace class, density matrix. In order to do so one has to expand the state up to the fourth order in λ. Applying the same method used for the other orders, one obtains a density matrix of the form:

$$\rho_{AB} = \begin{pmatrix} 1-|E_A|^2-|E_B|^2-\alpha-\beta-X & 0 & 0 & M+\delta \\ 0 & |E_B|^2+\beta & \langle E_A|E_B\rangle+\Pi & 0 \\ 0 & \langle E_B|E_A\rangle+\Pi^* & |E_A|^2+\alpha & 0 \\ M^*+\delta^* & 0 & 0 & X \end{pmatrix} + \mathcal{O}(\lambda^6). \tag{A11}$$

At this point we have simply numerically verified that $\alpha, \beta, \delta, \Pi$ are way smaller than X, and can thus be neglected. In fact, the only term needed to recover the positivity and the trace 1 of the density matrix is $X = |E_A|^2|E_B|^2 + |\langle E_A|E_B\rangle|^2 + |M|^2$, so that the correct density matrix to use is given by:

$$\rho_{AB} = \begin{pmatrix} 1-|E_A|^2-|E_B|^2-X & 0 & 0 & M \\ 0 & |E_B|^2 & \langle E_A|E_B\rangle & 0 \\ 0 & \langle E_B|E_A\rangle+ & |E_A|^2 & 0 \\ M^* & 0 & 0 & X \end{pmatrix}. \tag{A12}$$

Appendix B: Three different acceleration scenarios

In this section we give the expression for the spacetime trajectories in the three different acceleration scenarios of the two detectors, i.e., *parallel* $a_\parallel$, *antiparallel* $a_{\nparallel}$ and *perpendicular* $a_\perp$, and show how the matrix elements of ρ_{AB} only depend on the acceleration in all three cases.

1. Parallel acceleration

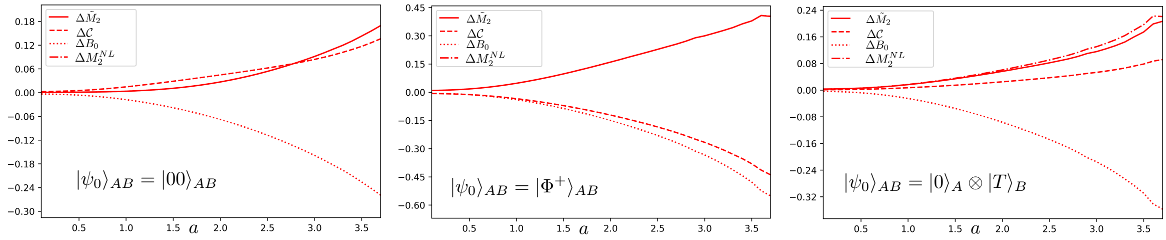
Suppose the two detectors are accelerated along the x-direction with acceleration $a_\parallel$ and initial mutual distance L fixed to 1. The trajectories of both detectors have the y ad z coordinates set to 0 along the motion, while the t and x coordinates are written in terms of the proper time as:

$$t_A = \frac{\sinh a_\parallel \tau_A}{a_\parallel}, \tag{B1}$$

$$x_A = \frac{L}{2} + \frac{\cosh a_\parallel \tau_A - 1}{a_\parallel}, \tag{B2}$$

$$t_B = \frac{\sinh a_\parallel \tau_B}{a_\parallel}, \tag{B3}$$

$$x_B = -\frac{L}{2} + \frac{\cosh a_\parallel \tau_B - 1}{a_\parallel}. \tag{B4}$$



(a) (b) (c)

Figure 2: Plots of the harvested magic (local and non local) and entanglement, together with the expectation value of the Bell operator, as a function of the acceleration a in the case of antiparallel acceleration $a_{\nparallel}$. Panel 2a shows the evolution of the harvested resources when the detectors start in the state $|\psi_0\rangle_{AB} = |00\rangle_{AB}$. Panel 2b shows the harvested resources when the detectors start in the Bell state $|\Phi^+\rangle_{AB}$. Panel 2c shows the harvested resources when the detectors are initialized in the state $|\psi_0\rangle_{AB} = |0\rangle_A \otimes |T\rangle_B$.

Let us look at the expression of the term $|E_A|^2$:

$$|E_A|^2 = \langle E_A^+|E_A^-\rangle = \int\int e^{-\frac{\tau_A}{2\sigma^2}} e^{-\frac{\tau_A'}{2\sigma^2}} e^{-i\Omega(\tau_A-\tau_A')} W(\tau_A, \tau_A', a_{\parallel}) d\tau_A d\tau_A' \tag{B5}$$

where

$$\begin{aligned} W(\tau_A, \tau_A', a_{\parallel}) &:= \langle 0|\, \phi_A(x(\tau_A), t(\tau_A))\phi_A(x(\tau_A'), t(\tau_A'))\, |0\rangle & \text{(B6)} \\ &= -\frac{1}{4\pi^2}\frac{1}{(t(\tau_A) - t(\tau_A') - i\epsilon)^2 - |x(\tau_A) - x(\tau_A')|^2}. & \text{(B7)} \end{aligned}$$

is the Wightman function of the massless scalar field:

This function depends on the proper times and on the acceleration according to the trajectories of the detectors. Finally, after the integrations in (B5), the term $|E_A|^2$ is only a function of the acceleration.

In a similar way, all the matrix elements can also be derived as integral functions depending on the acceleration parameter. For example, the off-diagonal term:

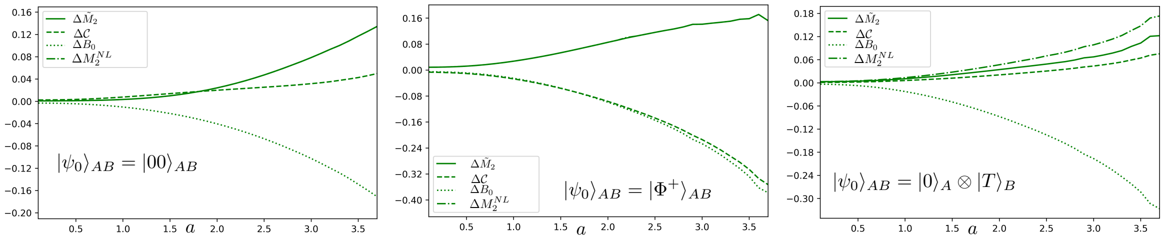
$$\langle E_A|E_B\rangle = \int\int e^{-\frac{\tau_A^2}{2\sigma^2}} e^{-\frac{\tau_B^2}{2\sigma^2}} e^{i\Omega(\tau_A+\tau_B)} W(\tau_A, \tau_B, a_{\parallel}) d\tau_A d\tau_B \tag{B8}$$

2. Anti-parallel acceleration

The setting for anti-parallel acceleration $a_{\nparallel}$ is the same as the previous case, but one of the two detectors has the spatial coordinate x with inverted sign:

$$\begin{aligned} t_A &= \frac{\sinh a_{\nparallel}\tau_A}{a_{\nparallel}}, & \text{(B9)} \\ x_A &= \frac{L}{2} + \frac{\cosh a_{\nparallel}\tau_A - 1}{a_{\nparallel}}, & \text{(B10)} \\ t_B &= \frac{\sinh a_{\nparallel}\tau_B}{a_{\nparallel}}, & \text{(B11)} \\ x_B &= -\frac{L}{2} - \frac{\cosh a_{\nparallel}\tau_B - 1}{a_{\nparallel}}. & \text{(B12)} \end{aligned}$$

With the same calculations shown for the parallel case, all the matrix elements of the detectors final state are written as integrals of the Wightman functions and after numerical integration over the proper times, all the terms will functions of the acceleration $a_{\nparallel}$. In Fig. 2, we show the harvested magic and entanglement together with the expectation value of the Bell operator for detectors with antiparallel acceleration.



(a) (b) (c)

Figure 3: Plots of the harvested magic (local and non local) and entanglement, together with the expectation value of the Bell operator, as a function of the acceleration $a_\perp$ in the case of perpendicular acceleration. Panel 3a shows the evolution of the harvested resources when the detectors start in the state $|\psi_0\rangle_{AB} = |00\rangle_{AB}$. Panel 3b shows the harvested resources when the detectors start in the Bell state $|\Phi^+\rangle_{AB}$. Panel 3c shows the harvested resources when the detectors are initialized in the state $|\psi_0\rangle_{AB} = |0\rangle_A \otimes |T\rangle_B$.

3. Perpendicular acceleration

Without loss of generality, we can take spacetime trajectories of detectors in perpendicular acceleration $a_\perp$ as:

$$t_A = \frac{\sinh a_\perp \tau_A}{a_\perp}, \tag{B13}$$

$$y_A = \frac{\cosh a_\perp \tau_A - 1}{a_\perp}, \tag{B14}$$

$$t_B = \frac{\sinh a_\perp \tau_B}{a_\perp}, \tag{B15}$$

$$x_B = \frac{\cosh a_\perp \tau_B - 1}{a_\perp} + L. \tag{B16}$$

In Fig. 3, we show the harvested magic and entanglement together with the expectation value of the Bell operator for detectors with perpendicular acceleration.