Universal properties of 2D statistical models at the critical point: a Conformal Field Theory approach

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Abstract

We will describe the Conformal Field Theory approach to the study of universal properties of 2D statistical models at the critical point. In particular we will show how different observables can be related to the representation theory of infinite Lie algrebra - the Virasoro algebra. We will focus on the 2D Ising model and implement the Transfer Matrix method to compute the free energy, critical exponents and the three-point correlation functions. We will match these findings to the CFT predictions.

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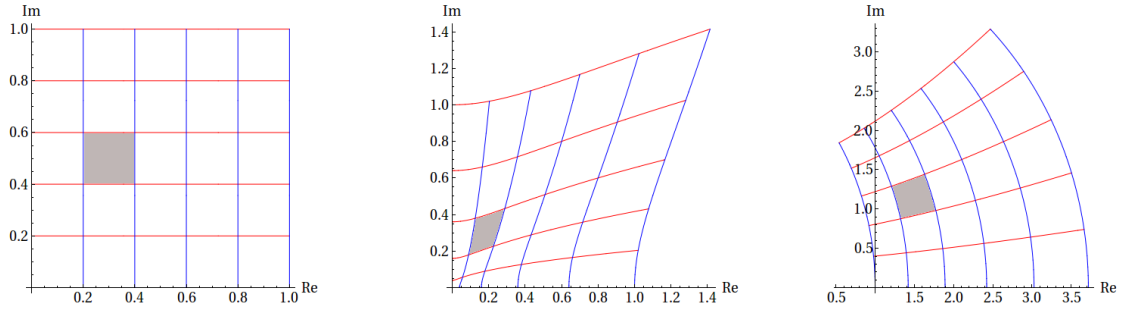

Figure 1: Coordinate transformations. (a) a square lattice in z (b) non-conformal lattice with a map $w(z) = |z|z$ (c) conformal lattice with a map $w(z) = z + e^z$

1 Introduction to Conformal Invariance

Close to the critical point, the correlation length of statistical systems diverges and fluctuations are present on all scales. At these critical points the universal properties of the system are expected to be captured by a quantum field theory. This QFT is scale invariant (massless) which means that the statistical averages are the same at any length scale. Implicitly it is assumed that such theories are also conformally invariant. Very recently this statment, that was taken as a theorem before, has been given an explicit proof by Smirnov [1] for statistical models like the Ising model and percolation for which he has been awarded a Fields medal in 2010. The ideas of conformally invariant theory were first put forward by A. Polyakov [2], and are continuing to be discussed in the context of numerous application of such theories in critical phenomena, particle physics, cosmology, gravitational physics and string theory (Nakamuara [3]).

Following chapters will serve as an introduction to conformal invariance and its implications. They are based on CFT manuals of Ginsparg [4], Henkel [5], Di Francesco [6] and Blumenhagen [7].

1.1 Conformal Transformation in d Dimensions

Let us begin by defining conformal transformation in d dimensions. Consider a line element in a space with metric $g_{\mu\nu}$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

Under an arbitrary local change of coordinates $x^\mu \longrightarrow x'^\mu$ the metric changes covariantly:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x^\mu) \tag{2}$$

Now, *conformal transformation* are defined as coordinate transformations which keep the metric invariant, up to a local scale factor

$$g'_{\mu\nu}(x'^\mu) = \Omega(x^\mu) g_{\mu\nu}(x^\mu) \tag{3}$$

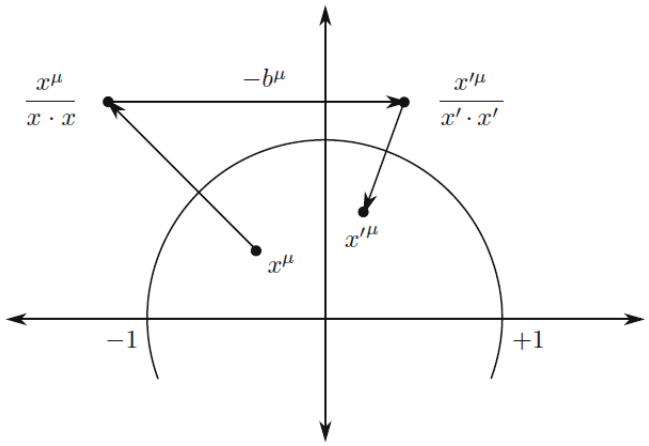

Figure 2: Illustration of a Special Conformal Transformation

Note that for flat spaces the scale factor $\Omega(x) = 1$ corresponds to the Poincaré group (if $g_{\mu\nu} = (-1, 1, 1, 1)$)consisting of translations and rotations, respectively Lorentz transformation. Now, if we define $\mathbf{x} \cdot \mathbf{x}' = g_{\mu\nu} x^\mu x'^\nu$ we are led to conclude that the angle between two vectors is preserved

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{x}'}{\sqrt{x^2 x'^2}} \longrightarrow \frac{\Omega\mathbf{x}\cdot\mathbf{x}'}{\Omega\sqrt{x^2 x'^2}} \tag{4}$$

Geometrically, this is illustrated in Fig. 1.

The set of conformal transformations forms a group, the *conformal group*. In $d > 2$ dimensions, the conformal group is finite-dimensional. It is generated by global *translations* $x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu$, global *dilatations* $x^\mu \longrightarrow x'^\mu = \alpha x^\mu$, global *rotations* $x^\mu \longrightarrow x'^\mu = M^\mu_\nu x^\nu$, and the so-called *Special Conformal Transformations*

$$x^\mu \longrightarrow x'^\mu = \frac{x^\mu + (x\cdot x)b^\mu}{1 + 2(b\cdot x) + (b\cdot b)(x\cdot x)} \tag{5}$$

The geometric meaning of the special transformation becomes clear when rewriting it as:

$$\frac{x'^\mu}{x'\cdot x'} = \frac{x^\mu}{x\cdot x} - b^\mu \tag{6}$$

which is the combination of an inversion $x^\mu \longrightarrow x'^\mu = x^\mu/x^2$ followed by a translation $-b^\mu$ and again an inversion as in Fig. 2.

It is natural to study the infinitesimal coordinate transformations which, up to first order in a small parameter $\epsilon(x) \ll 1$, read

$$x'^\alpha = x^\alpha + \epsilon^\alpha(x) \tag{7}$$

and we compute:

$$\begin{aligned} g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} &= g_{\alpha\beta}\Big(\delta^\alpha_\mu + \frac{\partial\epsilon^\alpha}{\partial x^\mu}\Big)\Big(\delta^\nu_\nu + \frac{\partial\epsilon^\beta}{\partial x^\nu}\Big) \\ &= g_{\mu\nu} + (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \end{aligned}$$

The requirement that the transformation be conformal implies that

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\nu = f(x)g_{\mu\nu} \tag{8}$$

The factor $f(x)$ is determined by tracing the equation on both sides with $g^{\mu\nu}$:

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho \tag{9}$$

Using this expression and solving for $f(x)$, we find the following restriction on the transformation Eq. 7 to be conformal

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}} \tag{10}$$

For simplicity, we assume that the conformal transformation is an infinitesimal deformation of the standard Cartesian metric $g_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu} = \text{diag}(1, 1, ..., 1)$. By applying an extra derivative ∂_ρ on Eq. (8), permuting the indices and taking a linear combination, we arrive at

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\eta f \tag{11}$$

Upon contacting with $\eta^{\mu\nu}$, this becomes

$$2\partial^2 \epsilon_\mu = (2 - d)\partial_\mu f \tag{12}$$

Applying ∂_ν on this expression and ∂^2 on Eq. (8), we find

$$(2 - d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f \tag{13}$$

Finally, contracting with $\eta^{\mu\nu}$, we end up with

$$(d - 1)\partial^2 f = 0 \tag{14}$$

First, if $d = 1$, the above equations do not impose any constraint on the function f, and therefore any smooth transformation is conformal in one dimension. This is a trivial statement, since the notion of angle then does not exist. The case $d = 2$ will be studied in the following chapter (Ch. 1.2). For the case $d \geq 3$ one can verify that translation , rotation, dilatation and the special transformation are solutions of Eq. 10 of order zero, one and two, respectively. One can also verify that the given examples of transformation exhaust all possibilities for conformal transformation for $d \geq 3$. (Ref. [7], [6])

For each of the infinitesimal transformation one can identify a corresponding generator(Table 1).These generators obey the following commutation rules, which in fact define the conformal algebra

$$\begin{aligned}
\big[D, P_\mu\big] &= iP_\mu \\
\big[D, K_\mu\big] &= -iK_\mu \\
\big[K_\mu, P_\nu\big] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
\big[K_\rho, L_{\mu\nu}\big] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
\big[P_\rho, L_{\mu\nu}\big] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
\big[L_{\mu\nu}, L_{\rho\sigma}\big] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})
\end{aligned}$$

	Transformations	Generators
translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
dilatation	$x'^{\mu} = \alpha x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
rotation	$x'^{\mu} = M^{\mu}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
SCT	$x'^{\mu} = \dfrac{x^{\mu} + (x\cdot x)b^{\mu}}{1 + 2(b\cdot x) + (b\cdot b)(x\cdot x)}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu})$

Table 1: Finite conformal transformations and corresponding generators

1.2 Conformal Transformation in Two Dimensions

In $d = 2$ we can adopt complex coordinates z ,$\overline{z}$:

$$\begin{aligned} z &= x + iy \quad \epsilon(z) = \epsilon_x + i\epsilon_y \\ \overline{z} &= x - iy \quad \overline{\epsilon}(\overline{z}) = \epsilon_x - i\epsilon_y \end{aligned}$$

0In these coordinates Eq. 10 takes the form of Cauchy-Riemann (CR) equations:

$$\begin{aligned} \partial_{\overline{z}}\epsilon &= 0 \\ \partial_z\overline{\epsilon} &= 0 \end{aligned}$$

Functions that satisfy the CR equations are all the analytic (holomorphic) mappings. Any such function can be written down as Laurent series with infinite number of parameters around $z = 0$. Expressing such holomorphic functions as an infinitesimal transformation we have:

$$z' = z + \epsilon(z) \quad \text{with} \quad \epsilon(z) = \sum_{n=-1}^{\infty} c_n z^{n+1} \tag{15}$$

We are naturally interested in the generator algebra of such infinitesimal transformations and we can examine it by studying the effects of such a mapping on a scalar field $\phi(z, \overline{z})$ defined on a complex plane

$$\begin{aligned} \phi'(z', \overline{z}') &= \phi(z, \overline{z}) \\ &= \phi(z, z) - \epsilon(z')\partial'\phi(z, \overline{z}') - \overline{\epsilon}(\overline{z}')\overline{\partial}'\phi(z', \overline{z}') \end{aligned}$$

The infinitesimal variation of the field is then

$$\begin{aligned} \delta_{\epsilon}\phi &= -\epsilon(z)\partial\phi - \overline{\epsilon}\overline{\partial}\phi \\ &= \sum_n \left[c_n l_n \phi(z, \overline{z}) + \overline{c}_n \overline{l}_n \phi_{z,\overline{z}}\right] \end{aligned}$$

where we introduced the generators

$$l_n = -z^{n+1}\partial_z \quad , \quad \overline{l}_n = -\overline{z}^{n+1}\partial_{\overline{z}} \tag{16}$$

The generators admit the commutation relations:

$$\begin{aligned} \left[l_n, l_m\right] &= (n - m)l_{n+m} \\ \left[\overline{l}_n, \overline{l}_m\right] &= (n - m)\overline{l}_{n+m} \\ \left[l_n, \overline{l}_m\right] &= 0 \end{aligned}$$

Transformation	Generators
translation	$l_{-1}, \bar{l}_{-1}$
dilatation	$l_0 + \bar{l}_0$
rotation	$i(l_0 - \bar{l}_0)$
SCT	$l_1, \bar{l}_1$

Table 2: Subalgebra generators in $d = 2$

This algebra is called the Witt algebra. The algebra is a direct sum of two isomorphic algebras.

Each of these two infinite-dimensional algebras contains a finite subalgebra generated by l_{-1}, l_0, l_1 (Table 2). The set $\{l_{-1}, l_0, l_1\}$ generates the *global projective conformal transormation* or *Möbius transformation*.

$$w_p(z) = z' = \frac{az + b}{cz + d} \tag{17}$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$ (See Appendix A).

1.3 Primary and Quasi-Primary Fields

Now we will define the notion of *primary field*, which plays a fundamental role in CFT.It is an object that transforms under the conformal map $z \to w(z)$, $\overline{z} \to \overline{w}(\overline{z})$:

$$\phi(z, \overline{z}) \longrightarrow \tilde{\phi}(w(z), \overline{w}(\overline{z})) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}} \phi(z, \overline{z}) \tag{18}$$

where the ϕ is the field of our theory and h (respectively $\overline{h}$) are real parameters and are called holomorphic (respectively anti-holomorphic) conformal dimension of the field. The conformal dimensions are related to the spin s and the scaling dimension Δ of the field by $s = h - \overline{h}$ and $\Delta = h + \overline{h}$. If the fields transform covariantly only under the projective group Eq. 17 they are called *quasi-primary fields*. All primary fields are also quasi-primary, but the revers is not true. as we shall see, an example of a quasi-primary field that is not primary is the energy-momentum tensor. A field which is not primary is generally called *secondary* or *descendant* field.

1.4 Correlation functions

We define the correlation function of n primary fields ϕ_i with associated conformal dimensions h_i and $\overline{h}_i$ as

$$\langle \phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n) \rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}_i}_{\overline{w}=\overline{w}_i} \langle \phi_1(z_1, \overline{z}_1) \cdots \phi_n(z_n, \overline{z}_n) \rangle \tag{19}$$

This relation tells us how the correlation function changes under a conformal mapping. We procede to give the forms of the two-point and three-point correlation functions for global projective transformations.

1.4.1 Two-Point Functions

Let $\phi_i(z)$ be scalar quasi-primary fields. Due to rotational and translational invariance, the two-point function has to be a function of the absolute value of the distance $z_1 - z_2$

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = f(|z_1 - z_2|) \tag{20}$$

The invariance under scaling transformation $z \to \lambda z$ requires

$$f(|z_1 - z_2|) = \lambda^{h_1+h_2} f(\lambda|z_1 - z_2|) \tag{21}$$

This means that

$$f(\lambda z) = \frac{1}{\lambda^{h_1+h_2}} f(z)$$
$$f(\lambda) = \frac{1}{\lambda^{h_1+h_2}} f(1) =: \frac{C_{12}}{\lambda^{h_1+h_2}}$$

gives

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{C_{12}}{|z_1 - z_2|^{h_1+h_2}} \tag{22}$$

Where C_{12} is a constant coefficient. In addition to scaling, translational and rotational invariance, it remains to introduce the invariance under special conformal transformations. For such transformation the Jacobian is:

$$\left|\frac{dw}{dz}\right| = \frac{1}{1 + 2(b \cdot z) + b^2 z^2} \tag{23}$$

If we define $\gamma_i = 1 + 2(b \cdot z_i) + b^2 z_i^2$, then the distance transformas as

$$\left|z_i - z_j\right| \longrightarrow \left|z_i' - x_j'\right| = \frac{\left|z_i - z_j\right|}{\sqrt{\gamma_i \gamma_j}} \tag{24}$$

If we then apply this to the correlation function

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \gamma_1^{-h_1}\gamma_2^{-h_2}\frac{C_{12}}{|z_1' - z_2'|^{h_1+h_2}}$$
$$= \frac{1}{\gamma_1^{h_1}\gamma_2^{h_2}}\frac{C_{12}(\gamma_1\gamma_2)^{\frac{h_1+h_2}{2}}}{|z_1 - z_2|^{h_1+h_2}}$$

$$\Longrightarrow \langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{(\gamma_1\gamma_2)^{\frac{h_1+h_2}{2}}}{\gamma^{h_1}\gamma^{h_2}}\langle \phi_1(z_1)\phi_2(z_2)\rangle$$

This relation is identically satisfied only if $h_1 = h_2 = h$. This implies that quasi-primary fields are correlated iff they have the same scaling dimension. With its anti-holomorphic part, the two-point function now reads

$$\langle \phi_1(z_1;\overline{z}_1)\phi_2(z_2,\overline{z}_2)\rangle = \delta_{h_1,h_2}\frac{C_{12}}{|z_1 - z_2|^{2h}|\overline{z}_1 - \overline{z}_2|^{2\overline{h}}} \tag{25}$$

where C_{12} is not a free parameter left to be determined by conformal invariance, but simply a normalization factor.

1.4.2 Three-Point Functions

The structure of the three-point functions can be obtained in the same way. We will write $z_{ij} = z_i - z_j$ as a shorthand notation to designate the distance. First by demanding rotational and translational invariance, we have now

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(|z_{12}|, |z_{13}|, |z_{23}|) \tag{26}$$

and due to scaling invariance we have

$$f(x, y, z) = \lambda^{h_1+h_2+h_3} f(\lambda x, \lambda y, \lambda z) \tag{27}$$

where we used x, y, z as general designation for coordinates. This is correct iff

$$x^a y^b z^c = f(x, y, z)(\lambda x)^a (\lambda y)^b (\lambda z)^c f(\lambda x, \lambda y, \lambda z) \tag{28}$$

for some a, b, c with $a + b + c = h_1 + h_2 + h_3$. Thus the left hand side has to be a constant with respect to x, y, z and we end up with

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}^{abc}}{|z_{12}|^a |z_{13}|^b |z_{23}|^a} \tag{29}$$

If we impose the special conformal transformation

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \gamma_1^{-h_1}\gamma_2^{-h_2}\gamma_3^{-h_3} \frac{C_{123}^{abc}}{(\gamma_1\gamma_2)^{-a/2}|z_{12}|^a(\gamma_1\gamma_3)^{-b/2}|z_{13}|^b(\gamma_2\gamma_3)^{-c/2}|z_{23}|^c}$$

$$= \gamma_1^{-h_1+\frac{a+b}{2}}\gamma_2^{-h_2+\frac{a+c}{2}}\gamma_3^{-h_3+\frac{b+c}{2}} \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle$$

Again like in the two-point correlation case, this relation must be identically satisfied, which produces the following relation:

$$2h_1 = a + b \qquad 2h_2 = a + c \qquad 2h_3 = b + c \tag{30}$$

which is set of linear equations that can be easily solved producing

$$\begin{aligned} a &= h_1 + h_2 - h_3 \\ b &= h_1 + h_3 - h_2 \\ c &= h_2 + h_3 - h_1 \end{aligned}$$

This gives us a nice form for the three-point correlation function

$$\langle \phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2)\phi_3(z_3, \overline{z}_3)\rangle = \frac{C_{123}}{(|z_{12}|^a |z_{13}|^b |z_{23}|^c)(|\overline{z}_{12}|^{\overline{a}} |\overline{z}_{13}|^{\overline{b}} |\overline{z}_{23}|^{\overline{c}})} \tag{31}$$

It is important to note that the value of the coefficient C_{123} is not arbitrary. We will in later chapters identify this quantity as the operator product expansion (OPE) coefficient and ultimately estimate its values through numerical means.

1.5 Virasoro Algebra

Witt Algebra (Chap. 1.2) admits a so-called *central extension*. Without providing a mathematically rigorous definition, we state that the central extension $\tilde{g} = g \oplus \mathbb{C}$ of a Lie algebra $\tilde{g}$ by $\mathbb{C}$ is characterised by the commutation relations:

$$\begin{aligned} [\tilde{x}, \tilde{y}]_{\tilde{g}} &= [x, y]_g + cp(x, y) && \tilde{x}, \tilde{y} \in \tilde{g} \\ [\tilde{x}, c]_{\tilde{g}} &= 0 && x, y \in g \\ [c, c]_{\tilde{g}} &= 0 && c \in \mathbb{C} \end{aligned}$$

where $p : g \times g \longrightarrow \mathbb{C}$ is bilinear. Central extensions of algebras are closely related to projective representations which are common to Quantum Mechanics[1]. In the following, we are going to allow for such additional structure. More concretely, let us denote the elements of the central extension of the Witt algebra by L_n with $n \in \mathbb{Z}$ and write their commutation relations as

$$\big[L_m, L_n\big] = (m - n)L_{m+n} + cp(m, n) \tag{32}$$

The precise form of $p(m, n)$ is determined in the following way:

1. First, from Eq. (32) it is clear that $p(m, n) = -p(n, m)$ in order for $p(m, n)$ to be compatible with the anti-symmetry of the Lie bracket.

2. We also observe that one can always arrange for $p(1, -1) = 0$ and $p(n, 0) = 0$ by a redefinition

$$\hat{L_n} = L_n + \frac{cp(n, 0)}{n} \qquad \text{for} \quad n \neq 0 \qquad \hat{L_0} = L_0 + \frac{cp(1, -1)}{2}$$

   Indeed, for the modified generators we see that the $p(n, m)$ vanishes

$$\begin{aligned} \big[\hat{L_n}, \hat{L_0}\big] &= nL_n + cp(n, 0) = n\hat{L_n}, \\ \big[\hat{L_1}, \hat{L_{-1}}\big] &= 2L_0 + cp(1, -1) = 2\hat{L_0} \end{aligned}$$

3. Next, we compute the following particular Jacobi identity:

$$\begin{aligned} \big[\big[L_m, L_n\big], L_0\big] + \big[\big[L_n, L_0\big], L_m\big] + \big[\big[L_0, L_m\big], L_n\big] &= 0 \\ (m - n)cp(m + n, 0) + ncp(n, m) - mcp(m, n) &= 0 \\ (m + n)p(n, m) &= 0 \end{aligned}$$

   from which we infer that in the case $n \neq -m$ we have $p(n, m) = 0$. Therefore, the only non-vanishing central extensions are $p(n, -n)$ for $|n| \geq 2$

[1] More details can be found in Ref. [13] Ch. 3

4. We finally calculate the following Jacobi identity:

$$\big[\big[L_{-n}, L_n\big], L_{-1}\big] + \big[\big[L_n, L_{-1}\big], L_{-n+1}\big] + \big[\big[L_{-1}, L_{-n+1}\big], L_n\big] = 0$$
$$(-2n+1)cp(1,-1) + (n+1)cp(n-1,-n+1) + (n-2)cp(-n,n) = 0$$

which leads to a recursion relation of the form

$$p(n,-n) = \frac{n+1}{n-2}p(n-1,-n+1)$$

but note that given in the following form the relation is invariant under the shift of n (e.g. $n \to n-1$):

$$\frac{P(n,-n)}{n(n+1)(n-1)} = \frac{P(n-1,-(n-1))}{n(n-1)(n-2)}$$

so in particular

$$\frac{P(n,-n)}{n(n+1)(n-1)} = \frac{P(2,-2)}{6}$$

which leads to

$$P(n,-n) = \frac{n(n^2-1)}{6}P(2,-2) \qquad \forall n$$

where we can normalise $P(2,-2) = \frac{1}{2}$ [2]

The central extension of the Witt algebra is called the **Virasoro algebra** and the constant c is called the **central charge**.

$$\boxed{\begin{aligned}\big[L_m, L_n\big] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}\\ \big[\overline{L}_m, \overline{L}_n\big] &= (m-n)\overline{L}_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}\\ \big[\overline{L}_n, L_m\big] &= 0\end{aligned}} \tag{33}$$

One can repeat the same operations for the conjugate algebra $\overline{L}_n$ and arrive to the relation above. Also, the commutation relation are easily verified.

Witt algebra alone does not account for the central charge, which has important physical consequences. In later chapters we will see how Virasoro algebra comes out naturally in our theory, and therefore justify the mathematical exercise of the central extension.

[2] This normalisation is chosen such that the constant c has a particular value for the standard example of the free boson.

2 Conformal Ward identities and correlation functions

The quantum description of a physical system may be tackled using different methods. One of them, the *operator formalism*, consists in replacing classical quantities by operators acting on a vector space in which the states of the system reside. Another methods is called *path integration* or *functional integration*, and can be related to the first one (see Ref. [15]). We will work, for our current purposes, in the second formalism, in which a correlation function of $\Phi[\phi_i]$ for fields $\phi_i(z,\overline{z})$ reads

$$\langle \Phi_1 \Phi_2 ... \Phi_N \rangle = \frac{1}{Z} \int \mathcal{D}\phi \Phi_1 \Phi_2 ... \Phi_N e^{-S[\phi]} \tag{34}$$

where $Z = \int \mathcal{D}\phi e^{-S[\phi]}$, $\mathcal{D}\phi$ denotes the functional measure, and $S[\phi]$ is the action.

Classically, the invariance of the action under a continuous symmetry implies the existance of a conserved current via Noether's theorem. At the quantum level, a continuous summetry of the action and of the functional measure leads to constraints on the correlation functions, which are expressed via the so-called *Ward identities*.

2.1 Ward identities

In order to examine the consequences of the conformal symmetry on correlation function we preform is an infinitesimal transformation $w(z) = z + \epsilon(z)$,$\overline{w}(\overline{z}) = \overline{z} + \overline{\alpha}(\overline{z})$ from which the fields transform as:

$$\tilde{\Phi}(z,\overline{z}) = \big(1 + \partial_z \epsilon\big)^h \big(1 + \partial_{\overline{z}} \overline{\epsilon}\big)^{\overline{h}} \Phi(z + \epsilon, \overline{z} + \overline{\epsilon}) \tag{35}$$

If we develop this and keep just the linear terms in ϵ we find the corresponding variation as:

$$\delta_\epsilon \Phi(z,\overline{z}) \equiv \tilde{\Phi}(z,\overline{z}) - \Phi(z,\overline{z}) \approx (h\partial_z \epsilon + \epsilon \partial_z + \overline{h} \partial_{\overline{z}} \overline{\epsilon} + \overline{\epsilon} \partial_{\overline{z}}) \tag{36}$$

The general idea is as follows: a symmetry transformation can be seen as change of the fundamental degrees of freedom of the fields $\phi \longrightarrow \tilde{\phi}$, in terms of which the field theory is described. Performing the variation on the integral Eq. 119, will produce variation in the action $S[\phi]$ as well as in the fields $\Phi_1 \Phi_2 ... \Phi_N$, with the measure $\mathcal{D}\phi = \mathcal{D}\tilde{\phi}$. The sum of these two variation contributions must be zero:

$$\delta_\epsilon \langle \Phi_1 \Phi_2 ... \Phi_3 \rangle + \langle \delta S \Phi_1 \Phi_2 ... \Phi_N \rangle = 0 \tag{37}$$

where stands $\delta_\epsilon \langle \Phi_1 \Phi_2 ... \Phi_N \rangle$ for $\langle (\delta_\epsilon \Phi_1) \Phi_2 ... \Phi_N \rangle + ... + \langle \Phi_1 \Phi_2 ... (\delta_\epsilon \Phi_N) \rangle$ and the variation of the action term inside the correlation function reads [3]:

$$\langle \delta S \Phi_1 \Phi_2 ... \Phi_N \rangle = -\frac{1}{2\pi} \oint_C ds^\mu \epsilon^\nu \langle T_{\mu\nu} \Phi_1 \Phi_2 ... \Phi_N \rangle \tag{38}$$

where integration is performed in domain C and $T_{\mu\nu}$ is the stress-energy tensor. In complex coordinates the stress-energy tensor reads: $T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})$, $T_{\overline{z}\overline{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})$

In this way one finds:

[3] Exact derivation can be found in Dotsenko [9]

$$\frac{1}{2\pi i}\oint_C dw\Big[\epsilon\langle T_{zz}(w,\overline{w})\Phi_1\Phi_2...\Phi_N\rangle + \overline{\epsilon}\langle T_{\overline{z}z}(w,\overline{w})\Phi_1\Phi_2...\Phi_N\rangle\Big]$$
$$-\frac{1}{2\pi i}\oint_C d\overline{w}\Big[\overline{\epsilon}\langle T_{\overline{z}\overline{z}}(w,\overline{w})\Phi_1\Phi_2...\Phi_N\rangle + \epsilon\langle T_{z\overline{z}}(w,\overline{w})\Phi_1\Phi_2...\Phi_N\rangle\Big]$$
$$=\sum_{i=1}^{N}\Big[h_i\partial_z\epsilon(z_i)+\epsilon(z_i)\partial_{z_i}+\overline{h}_i\partial_{\overline{z}}\overline{\epsilon}(z_i)+\overline{\epsilon(z_i)}\partial_{\overline{z}_i}\Big]\langle\Phi_1\Phi_2...\Phi_N\rangle$$

Previous equation is the first version of the Ward identity, but in an integral form. We can further simplify it, and write it down in its local form. One can achieve this by considering two special cases $\alpha(z)=a$ and $\alpha(z)=bz$, where a,b are constant parameters. It can be shown through the use of the complex Stokes theorem that these cases correspond to

$$\partial_{\overline{z}}\langle T_{zz}\Phi_1\Phi_2...\Phi_N\rangle = 0 \tag{39}$$

$$\langle T_{z\overline{z}}\Phi_1\Phi_2...\Phi_N\rangle = 0 \tag{40}$$

Eq. 39 shows that inside correlation function T_{zz} does not depend on $\overline{z}$. Which means that T_{zz} is an *holomorphic operator*. Easily enough one can repeat the same argument and show that $T_{\overline{z}\overline{z}}$ is an *anti-holomorpic operator*. Hence we set

$$T_{zz}=T(z) \quad , \quad T_{\overline{z}\overline{z}}=\overline{T}(\overline{z}) \tag{41}$$

Also, given that $T^{\mu}_{\mu}=g^{\mu\nu}T_{\mu\nu}=T_{00}+T_{11}=\frac{1}{2}(T_{z\overline{z}}+T_{\overline{z}z})$ (metric being $g^{\mu\nu}=\mathrm{diag}(+1,+1)$), we see that Eq. 40 expresses the vanishing of the trace of the energy-momentum tensor inside a correlation function. The tracelessness of the energy-momentum tensor is sufficient to guarantee the invariance of the action under conformal transformation. Expressions for the tensors in the complex plane are given by:

$$T(z)=\frac{1}{2}\big(T_{00}-iT_{11}\big) \quad , \quad T(\overline{z})=\frac{1}{2}\big(T_{00}+iT_{11}\big) \tag{42}$$

In light of the previous simplifications, the Ward identity reduces to

$$-\frac{1}{2\pi i}\oint_C dw\epsilon\langle T\Phi_1\Phi_2...\Phi_N\rangle = -\sum_{i=1}^{N}\Big[h_i\partial_z\epsilon(z_i)+\epsilon(z_i)\partial_{z_i}\Big]\langle\Phi_1\Phi_2...\Phi_N\rangle \tag{43}$$

Using the residue theorem (see Appendix**??**) r.h.s can be expressed as a of contour integrals

$$\sum_{i=1}^{N}\Big[h_i\partial_z\epsilon(z_i)+\alpha(z_i)\partial_{z_i}\Big]\langle\Phi_1\Phi_2...\Phi_N\rangle = \sum_{i=1}^{N}\frac{1}{2\pi i}\oint_{C_i}dw\epsilon\Big[\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\Big]\Phi_1\Phi_2...\Phi_N$$

This is because α is analytic in all the contours C_i. Contours can be deformed to be enclosed in contour C, which embeds all the singular points Fig. 8

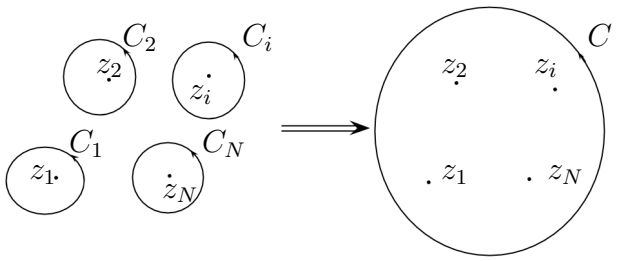

Figure 3: Enclosing of $i = 1, 2, ...N$ poles of Eq. 44 by the contour C

$$\frac{1}{2\pi i}\oint\limits_C dw\epsilon\langle T\Phi_1\Phi_2...\Phi_N\rangle = \frac{1}{2\pi i}\oint\limits_C dw\epsilon\sum_{i=1}^{N}\Big[\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\Big]\langle\Phi_1\Phi_2...\Phi_N\rangle \tag{44}$$

This is an equation of the from $\oint\limits_C dz\alpha(z)f(z) = 0$ for all analytic functions $\alpha(z)$

$$f(z) = \langle T(z)\Phi_1\Phi_2...\Phi_N\rangle - \sum_{i=1}^{N}\Big[\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\Big]\langle\Phi_1\Phi_2...\Phi_N\rangle \tag{45}$$

The requirement that the integral over $f(z)$ is zero immediatly implies that $f(z) = 0$. This way we obtain the local form of the Ward identities for the correlation:

$$\boxed{\langle T(z)\Phi_1\Phi_2...\Phi_N\rangle = \sum_{i=1}^{N}\Big[\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\Big]\langle\Phi_1\Phi_2...\Phi_N\rangle} \tag{46}$$

2.2 Operator Product expansion - OPE

We will use the fact that our theory decouple z from $\overline{z}$, and procede in our considerations only looking at the holomorphic part, but keeping in mind that everything is isomorphicly implied for the anti-holomorphic part. Eq. 45 yields the singular behaviour of the correlator of the field $T(z)$ with the primary fields $\Phi_i(w_i, \overline{w}_i)$ as $z \to w_i$. For a single primary field of conformal dimension h we can write

$$T(z)\Phi(w,\overline{w}) \sim \frac{h}{(z-w)^2}\Phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\Phi(w,\overline{w}) \tag{47}$$

This expression states that when $w \to z$ the correlation function $\langle T(z)\Phi(w,\overline{w})X\rangle$ behaves like a function $\frac{h}{(z-w)^2}\langle\Phi(w,\overline{w})X\rangle + \frac{1}{z-w}\partial_w\langle\Phi(w,\overline{w})X\rangle$, where $X = \Phi_1(w_1,\overline{w}_1)...\Phi_N(w_N,\overline{w}_{\overline{N}})$. An expression

like Eq. 47 is called (the singular part of) the *operator product expansion* (OPE) of the energy-momentum tensor with a primary field. It is important to note that OPE's make sence only inside a correlation function. The equality $\sim$ restrics the sum to non-regular or singular points.

Now an additional definition can be given for a primary operator, which states that a *primary* operator is one whose OPE with T truncates at order $(z-w)^{-2}$.

As an illustration the OPE of stress-energy tensor T can be shown to read:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{48}$$

which means that $T(z)$ is a quasi-primary field. c is the central charge as introduced in Chap. 1.5.

2.3 Variation of the fields

Now we will take a step-back and evaluate what an OPE implies. As we discussed in Chap. 2.1, in order to derivate the Ward identity we preformed the variation on the fields $\langle \Phi_1 \Phi_2 ... \Phi_N \rangle$. The variation is expressed as (taking the holomorphic part only):

$$\delta_\epsilon \langle \Phi_1 \Phi_2 ... \Phi_N \rangle = -\frac{1}{2\pi i} \oint_C dw \epsilon \langle T(w) \Phi_1 \Phi_2 ... \Phi_N \rangle \tag{49}$$

This equation is defining the transformation of the fields within the correlation function. We can further write it as

$$\begin{aligned} \delta_\epsilon \langle \Phi_1 \Phi_2 ... \Phi_N \rangle &\equiv \sum_i^N \langle \Phi_1 ... \delta_{\epsilon(z_i)} \Phi_i(z_i) ... \Phi_N \rangle \\ &= \sum_{i=1}^N \frac{1}{2\pi i} \oint_{C_i} dw \epsilon \langle T(w) \Phi_1 ... \Phi_N \rangle \end{aligned}$$

Each integral in the last expression corresponds to the variation of a field in equation from the previous line. Hence the variation of a single fields is represented by the integral

$$\boxed{\delta_{\epsilon(z_i)} \Phi(z_i) = \frac{1}{2\pi i} \oint_{C_i} dw \epsilon T(w) \Phi_i(z_i)} \tag{50}$$

where C_i encircles only one singular point z_i. Consequently, because ϵ is analytic inside C_i, we can conclude that the transformation of a field under $z \to z + \epsilon$ is completely determined by the singular terms of the OPE of this field with the holomorphic part T of the energy-momentum tensor, and vice-versa. This is why OPEs play an important role in CFT.

3 Representation Theory

Advantage of critical two-dimensional systems over other higher dimensional systems lies in the presence of the enlarged symmetry algebra given by the two infinite dimensional Virasoro algebras Eq. (33). This algebra take on a representation. In the proceding discussion we will focus on the holomorphic part of the algebra L_n,

$$\big[L_m, L_n\big] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{51}$$

but keep in mind that the full conformal algebra is generated by the tensor product of the algebras $L_n \otimes \overline{L}_m$.

3.1 Verma Module

The representations of the Virasoro algebra describe how the energy-momentum tensor acts on the physical scaling operators. For a primary operator ϕ, the essential information is contained in the conformal Ward identity (Eq. 46) and involves only the analytic component $T = T(z)$. We only need the singular part of this, which may be written down as the formal operator product expansion (Eq. 47)

$$T(z)\phi(w,\overline{w}) = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) + \text{regular terms...} \tag{52}$$

Virasoro generators L_n can be defined to be the Laurent expansion coeffients around the point w:

$$T(z)\phi(w,\overline{w}) = \sum_{n=-\infty}^{\infty}(z-w)^{-n-2}L_n\phi(w,\overline{w}) \tag{53}$$

and may be written, using Cauchy integral formula (Appendix **??**), as:

$$L_n\phi(w,\overline{w}) = \frac{1}{2\pi i}\oint_{C_w} dz(z-w)^{n+1}T(z)\phi(w,\overline{w}) \tag{54}$$

Comparing the leading terms in Eq. (52) and in Eq. (53), it follows that an algebraic definition of a primary operator ϕ can be formulated as

$$\boxed{\begin{array}{ll} L_n\phi = 0 & \text{if} \quad n \geq 1 \\ L_0\phi = h\phi & \end{array}} \tag{55}$$

In fact for ϕ to be primary, it is necessary and sufficient that

$$L_0\phi = h\phi \quad , \quad L_1\phi = 0 \quad , \quad L_2\phi = 0 \tag{56}$$

To see this, note that the other constraints $L_n\phi = 0$ for $n \geq 3$ can be deduced by recursion from Eq. (55). From Eq. (51) we have $[L_n, L_1] = (n-1)L_{n+1}$. now, by induction over n

$$L_{n+1}\phi = \frac{1}{n-1}(L_nL_1\phi - L_1L_n\phi) = 0 \tag{57}$$

follows from all $n \geq 3$, if it is valid for $n = 1, 2$. L_0 measures the holomorphic part of the conformal dimension of the operator. This holds for primary as well as for non-primary operators. Notice also that L_{-1} generates the translations. Hence we have the identification

$$\boxed{L_{-1} = \partial_z} \tag{58}$$

A primary operator defines the *highest weight* of a representation of the confromal symmetry. An infinite set of secondary (non-primary) or descendant operators are defined by the negatively indexed operators L_{-n}. They build up the *Verma module* of ϕ whose elements are given by

$$\phi^{(-n_k,...,-n_1)} = L_{-n_k}...L_{-n_1}\phi \tag{59}$$

The set of secondary (descendant) operators, obtained from ϕ, is referred to as the *conformal tower* of ϕ. The integer $n := n_1 + n_2 + ... + n_k$ is called *level* of the operator $\phi^{(-n_k,...,-n_1)}$. We impose the condition $n_1 \geq n_2 \geq ... \geq n_k$. Secondary operators are all eigenvectors of L_0 with eigenvaluesc $h + \sum_i n_i$, since

$$\begin{aligned} L_0\phi^{(-n_k,...,-n_1)} &= L_0 L_{-n_k}...L_{-n1}\phi \\ &= n_k L_{-n_k}...L_{-n_1}\phi + L_{-n_k}L_0...L_{-n_1}\phi \\ &= (n_k + ... + n_1 + h)\phi^{(-n_k,...,-n_1)} \end{aligned}$$

Primary operator ϕ is the only element of the Verma module with conformal weight h and $L_{-1}\phi$ the only one with confromal weight $h + 1$. At conformal weight $h + 2$, we have two elements, $L_{-2}\phi$ and $(L_{-1})^2\phi$.

level	dimension	state
0	Δ	ϕ
1	$\Delta + 1$	$\phi^{(-1)} = L_{-1}\phi$
2	$\Delta + 2$	$\phi^{(-1,-1)} = L_{-1}^2\phi$; $\phi^{(-2)} = L_{-2}\phi$
3	$\Delta + 3$	$\phi^{(-1,-1,-1)}$; $\phi^{(-1,-2)}$; $\phi^{(-3)}$
...	...	...
N	$\Delta + N$	$P(N)$ fields

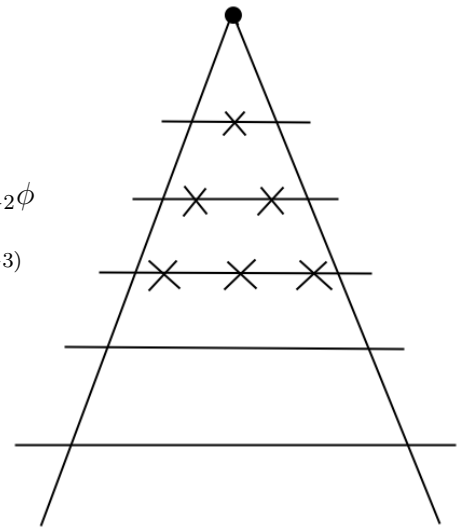

Figure 4: Conformal tower (Verma module) of a holomorphic heighest weight field ϕ. $P(N)$ is the number of partitions of N in integer parts.

The argument leading to the algebraic conditions Eq. 55 and 58 can be extended to non-primary operators. Considering the OPE of the stress-energy tensor (Eq. 48) together with the expansion Eq. (53) we read off for the coefficients:

$$L_0 T = 2T(z_1) \quad , \quad L_1 T = 0 \quad , \quad L_2 T = \frac{c}{2} \quad , \quad L_n T = 0 \quad ; \quad \text{for} n \geq 3 \tag{60}$$

Now if we would insert, as an important special case of a primary operator, the identity operator I into Eq. (53). Because of the regularity of the integral involved for $n \geq -1$, the identity operator is primary with $L_0 I = 0$ nad $L_{-1} I = 0$. However, $L_{-2}\mathbf{1}$ is non-vanishing

$$L_{-2} I = T(z) \tag{61}$$

The energy-momentum tensor T is therefore a secondary operator of the identity. Simple algebraic consideration: $L_1 T = L_1 L_{-2}\mathbf{1} = 3L_{-1}\mathbf{1} + L_{-2}L_1\mathbf{1} = 0$ verify this statement. We say that it belongs to the conformal tower of the identity! A defining algebraic condition for a quasi-primary reads

$$L_{-1}\phi = 0 \tag{62}$$

We give some examples of quasi-primary operators, stating that the highest weight ϕ is quasi-primary as well. At level $N = 2$, the most general operator in the conformal tower of ϕ is of the form $\phi^{(2)} = (aL_{-2} + bL_{-1}^2)\phi$ where a, b are constants. It is easy to verify (using Eq. 62)

$$\phi^{(2)} = \Big(L_{-2} - \frac{3}{2(2h+1)} L_{-1}L_{-1}\Big)\phi \tag{63}$$

which makes the $\phi^{(2)}$ a quasi-primary operator. At level $N = 3$, there are two derivative operators $L_{-1}L_{-1}L_{-1}\phi$, $L_{-1}\phi^{(2)}$, and the new quasi-primary operator is given as:

$$\phi^{(3)} = \Big(L_{-3} - \frac{2}{h+2} L_{-1}L_{-2} + \frac{1}{(h+2)(h+1)} L_{-1}L_{-1}L_{-1}\Big)\phi \tag{64}$$

Two quasi-primary operators appear at level $N = 4$, one containing $L_{-4}\phi$ and the other $L_{-2}L_{-2}\phi$. We observe that the number of operators increses with the level N. If we assume that the infinite dimensional representation, made out of the descendant operators $L_{-n_k}...L_{-n_1}\phi$ with $n_1 \geq n_2 \geq ... \geq n_k$, is irreducible, this would imply that all descendant operators at a given level N are linearly independant. The number of independant operators is the number of partitions $P(N)$ of N. We define the *generic character* of the Verma module by

$$\chi(h, c) = q^{h - \frac{c}{24}} \sum_{n=0}^{\infty} P(N) q^N = q^{h - \frac{c}{24}} \prod_{i=1}^{\infty} \frac{1}{1 - q^i} \tag{65}$$

The product in this equation is up to a factor $q^{1/24}$ is the inverse Dedekind's function $\eta(\tau)$. The character encodes the information on the degenerate structure of the Verma module. It is possible that the representation of the Virasoro algebra Eq. 9 is *reducible*. This means that there exist a subspace(submodule) that is in itself a representation of the Virasoro algebra. Such a submodule is also generated from a heighest-weight state $|\chi\rangle$[4] such that:

$$L_n|\chi\rangle = 0 \quad \text{when} \quad n > 0 \tag{66}$$

State annihilated by all $L_n (n > 0)$ is called *null vector*. Such a state generates its own Verma module included in the original module. Null vectors are orthogonal to the Verma module. These states decouple from our theory and are important in studying of higher order correlation functions.

[4]States such as $\phi^{(2)}$ and $\phi^{(3)}$ as an example

3.2 Unitarity and Kac determinant

In order to properly define representations, we have to specify on which states they should act. We define the vacuum state as the highest weight of the identity operator:

$$|0\rangle := I \tag{67}$$

This state is defined by $L_n|0\rangle = 0$ for $n \geq -1$. The true physical vacuum which recombines the z and $\overline{z}$ dependencies, is the tensor product $|0\rangle \otimes |\overline{0}\rangle$. A primary operator can now be viewed as a mapping between two vector spaces such that the vacuum state is mapped onto a highest weight state defined by:

$$|h\rangle := \phi_h|0\rangle \tag{68}$$

with properties $L_n|h\rangle = 0$ for $n > 0$ and $L_0|h\rangle = h|\rangle$. This way we have shown that field ϕ can be seen as an operator $\hat{\phi}$, and the transition $field \longrightarrow operator$ has been made.

For our purpose we are going to discuss *minimal models* which, unitary or not, are characterized by the fact that the number of distinct Verma modules is finite. *Unitarity* is characterized by the absence of negative-norm states in the theory. States at level $N = 0$ do not provide any information, since we can normalize $\langle h|h\rangle = 1$. At level $N = 1$ we have the state $L_{-1}|h\rangle$ with norm

$$\langle h|L_1 L_{-1}|h\rangle = 2h \tag{69}$$

Therefore, a necessary condition for unitarity is $\boxed{h \geq 0}$. Another simple condition is given by the two-point function of the energy-momentum tensor. The norm is given by

$$\langle 0|L_2 L_{-2}|0\rangle = \frac{c}{2} \tag{70}$$

Therefore $\boxed{c \geq 0}$ is also neccessary to ensure unitarity. To gain new information, we have to investigate higher levels in the Verma module. At level $N = 2$, we have two independent vectors $L_{-2}|h\rangle$ and $(L_{-1})^2|h\rangle$. The unitarity condition requires that the determinant of the matrix:

$$M_2(h,c) = \begin{pmatrix} \langle h|L_2 L_{-2}|h\rangle & \langle h|L_2 (L_{-1})^2|h\rangle \\ \langle h|(L_1)^2 L_{-2}|h\rangle & \langle h|(L_1)^2 (L_{-1})^2|h\rangle \end{pmatrix} = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \tag{71}$$

is positive definite. This is equivalent to the requirement that the matrix is positive definite, which is the case iff its eigenvalues are positive. An arbitrary choice of the conformal weight h and the central charge c may result in negative eigenvalues at level 2, therefore breaking unitarity. This means that the determinant of the matrix Eq. 71 amounts to:

$$\det M_2(c,h) = 32 h_{1,1}(h - h_{1,2})(h - h_{2,1}) \tag{72}$$

where the solutions are

$$\begin{aligned} h_{1,1} &= 0 \\ h_{1,2} &= \frac{1}{16}(5 - c - \sqrt{25 - 26c + c^2}) \\ h_{2,1} &= \frac{1}{16}(5 - c + \sqrt{25 - 26c + c^2}) \end{aligned}$$

Determinant changes sign when passing in the (c,h) plane one of the curves $h_{1,2}, h_{2,1}$. In Fig. 5 the two curves are displayed. The are called *vanishing curves*, because the determinant $\det M_2(c,h)$ vanishes along them. In the region in between the curves the determinant is negative, which makes this region less interesting for us.

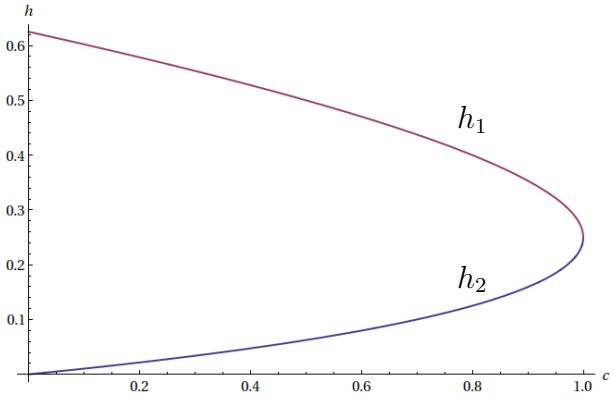

Figure 5: Vanishing curves for level $N = 2$

The determinant computed at level $N = 2$ can be generalized to any level. At level N, Kac found the following formula:

$$\det M_N(c,h) = t_N \prod_{r,s=1;1\leq rs\leq N}^{N} (h - h_{r,s})^{P(N-rs)} \tag{73}$$

where t_N is a positive constant. A particular parametrization was introduced to write this formula. The central charge is defined as

$$c := 1 - \frac{6}{m(m+1)} \tag{74}$$

where m is can be real or complex. A model is called *minimal* if m is a rational number. The zeros of the determinant take values:

$$h_{r,s} = h_{m-r,m+1-s} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)} \tag{75}$$

where r and s are positive integers and $P(N - rs)$ is the number of partitions of integer difference $N - rs$.

In Fig. 6 curves $h = h_{r,s}$ have been plotted for levels $N = 2, 3, 4$. For a given level N, the determinant $\det M_N(c, h)$ changes sign across the vanishing curves $h = h_{r,s}$, with $rs = N$. The points where the two vanishing curves intersect are of great importance to us. These points are called *first intersections*. First intersection occur for $c = 1/2$ at $h = 1/16$ and $h = 1/2$ and for $c = 7/10$ at $h = 3/2$. In these cases, we see from Eq.74 that $m = 3, 4$, respectivly.

The infinite sequence of unitary conformal models which correspond to value $m = 2, 3, 4, 5, ...$ is called *unitary minimal series*. For each value of m the table of possible conformal weights $h_{r,s}$ with $r = 1, ..., m - 1$ and $s = 1, ..., m$ is called the *Kac table* (Fig. 7).

First interesting case is when $m = 3$ where $c = 1/2$, shown in Fig 7. Here, we expect to have the vacuum representation and two non-trivial representations. There are $|h_{1,2}\rangle$ with $h_{1,2} = h_{2,2} = 1/16$

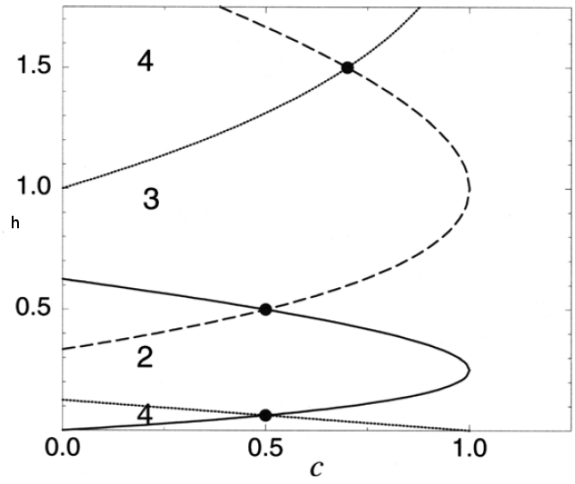

Figure 6: Vanishing curves $h = h_{r,s}$ for first four levels. The lines correspond to level 2 (full), level 3 (dashed) and level 4 (dotted). In the refions labeled 2,3,4 the determinant $\det M_N(c,h)$ is negative. The points mark the first intersections at $c = 1/2$ and $h = 1/16, 1/2$ and $c = 7/10$ and $h = 3/2$.

$\frac{1}{2}$	0	
$\frac{1}{16}$	$\frac{1}{16}$	
0	$\frac{1}{2}$	

$q \uparrow$

$p \rightarrow$

$\frac{3}{2}$	$\frac{7}{16}$	0
$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$
0	$\frac{7}{16}$	$\frac{3}{2}$

Figure 7: Kac table for first interection points. Corresponding to $m = 3, 4 (c = 1/2, 7/10)$

and $|h_{1,3}\rangle$ with $h_{1,3} = h_{2,1} = 1/2$. We may identify the $m = 3$ unitary model with the *Ising model* where $|h_{1,2}\rangle$ corresponds to the holomorphic part of the spin-density operator ($\Delta^\sigma = 1/8 = 2h_{1,2}$) and $|h_{1,3}\rangle$ represents the holomorphic part of the energy-density operator $\Delta^\epsilon = 1 = 2h_{1,3}$ The identification we make:

$$\boxed{I = \phi_{1,1} \quad , \quad \sigma = \phi_{1,2} = \phi_{2,2} \quad , \quad \epsilon = \phi_{2,1} = \phi_{1,3}} \tag{76}$$

Further identifications are possible. For $m = 4$, we find $c = 7/10$ which is the tri-critical Ising model (Fig. 7). The next intersection is the three-state Potts model.

3.2.1 Critical exponents

Now that we have showed what are the fields involved for our statistical system, we procede to compair them with the usual quantities we have in conventiaonal description of the Ising model on a lattice. The first one $\phi_{(1,1)}$ is present in every theory and is identified as the identity operator. Lets consider a the Ising model with the order parameter $\sigma = \pm 1$, . At the critical point the correlation length diverges (theory becomes massless) and the two-point correlation function falls off as a power law (which implies scale invariance and which in turn implies conformal invariance):

$$\langle \sigma_0 \sigma_r \rangle \sim \frac{1}{r^{d-2+\eta}} \tag{77}$$

where d is the the dimension of the system and this expression defines the critical exponent η. We have also the lattice interaction energy $\epsilon_i = \sum_k \sigma_i \sigma_{i+k}$, where k denotes the possible direction around the σ_i which at critical point behaves as:

$$\langle \epsilon_0 \epsilon_R \rangle \sim \frac{1}{r^{2(d-1/\nu)}} \tag{78}$$

where ν is related to the correlation length divergence (at $T \longrightarrow T_c$) expressed in terms of the reduced temperature $t = (T - T_c)/T_c$ so that

$$\xi \sim t^{-\nu} \tag{79}$$

For $d = 2$ the critical exponents calculated for the Ising model are $\eta = 1/4$, $\nu = 1$. This means that the two-point function has the following behaviour:

$$\langle \sigma_0 \sigma_r \rangle \sim \frac{1}{r^{1/4}} \sim \frac{1}{|x_1 - x_2|^{\Delta^\sigma}} \tag{80}$$

The superscript σ denotes that it is a spin correlations. According to Eq. 25 we recongnize that because of the condition $h_1^\sigma = h_2^\sigma$ which gives $\Delta^\sigma = 1/8$, we are able to identify the $\sigma = \phi_{2,2} = \phi_{(\frac{1}{16},\frac{1}{16})}$ field with the spin σ of the Ising model.

The energy operator, similarly complies to

$$\langle \epsilon_0 \epsilon_r \rangle \sim \frac{1}{r^{\Delta^\epsilon}} \tag{81}$$

The superscript ϵ denotes that it is a spin correlations. Now the we have $\Delta^\epsilon = 1$. Thus we identify field $\epsilon = \phi_{1,3} = \phi_{(\frac{1}{2},\frac{1}{2})}$ as being the energy operator of the Ising model. We should also mention that these field with these conformal weights are spinless.

This completes the identification of the three existing fields in the Ising model.

4 Ising model in 2D

Before analysing the consequances of conformal symmetry we give a general remainders on the 2D Ising model on a square lattice, and the Transfer Matrix method. The model is defined by Hamiltonian:

$$H(\{\sigma_{k,j}\}) = -J \sum_{j,k} \sigma_{k,j}\sigma_{k,j+1} + \sigma_{k,j}\sigma_{k+1,j} \tag{82}$$

where J is the coupling constant, and spin variable σ can take two values: $+1$ or -1. The spin variables have two indices corresponding to rows and columns of the square lattice, k and j respectively. We impose boundary conditions:

$$\sigma_{k,L} = \sigma_{k,1} \quad , \quad \sigma_{N,j} = \sigma_{1,j} \tag{83}$$

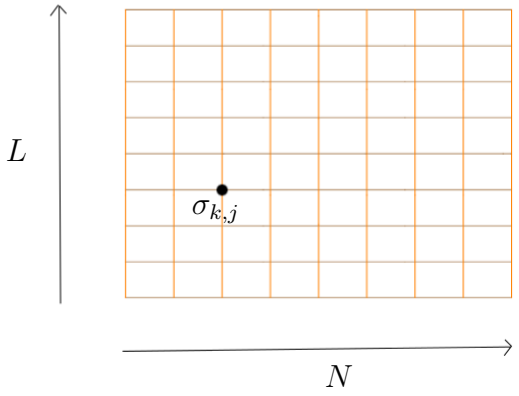

Figure 8: Square lattice

We procede to solve the Ising model, by the Transfer Matrix method. We will numerically evaluate quantities of interest (free energy, critical exponents) and compair with the CFT predictions. (Chap. 6)

4.1 Transfer Matrix Method

Transfer Matrix Method is potent way to solve statistical models, which is the analogue in statistical mechanics of the operator formalism in quantum field theory.

The partition function of the model is given by

$$Z(L,N) = \sum_{\{\sigma_{j,k}\}} e^{-\beta H(\{\sigma_{k,j}\})} \tag{84}$$

where $\beta = 1/T$, and T us the temperature. The idea of the TM approach is to write this in terms of matrix multiplications. We start by expressing the partition function in the form

$$Z(L,N) = \sum_{\{\sigma_{j,k}\}} e^{-\beta \sum_{k=1}^{N} E(k;k+1)} \tag{85}$$

where

$$E(k;k+1) = -J \sum_{j=1}^{N} \sigma_{k,j}\sigma_{k+1,j} + \frac{1}{2}\big[\sigma_{k,j}\sigma_{k,j+1} + \sigma_{k+1,j}\sigma_{k+1,j+1}\big] \tag{86}$$

This is nothing more then rewriting the Hamiltonian in a more symmetric way (Fig. 16).

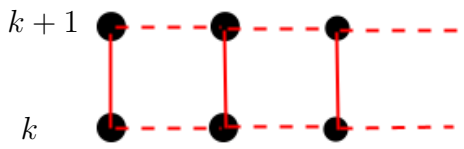

Figure 9: Symmetric interaction between spins in 2D Ising model; full lines J, dashed lines $J/2$

Now the Hamiltonian or the energy depends only on the configurations of spins on rows k and $k+1$, that is on spins $\sigma_{k,1}...\sigma_{k,L}$ and $\sigma_{k+1,1}...\sigma_{k+1,L}$. Each configuration of spins on a given row specifies a sequence $\sigma_1, \sigma_2, ..., \sigma_N$ with $\sigma_j = \pm 1$. Let us associate a vector

$$|\mu_k\rangle \tag{87}$$

with each such sequence. By construction there are 2^N such vectors. The vectors $\{|\mu\rangle\}$ form an orthonormal basis of a 2^N dimensional linear vector space. On this space, we define the action of the Transfer Matrix T of size $2^N \times 2^N$ by its matrix elements:

$$\langle \mu_k | T | \mu_{k+1} \rangle = e^{-\beta E(k;k+1)} \tag{88}$$

The point of this construction is that the partition function can now be written in the form:

$$Z = \sum_{\mu_1}\sum_{\mu_2}...\sum_{\mu_N} \langle \mu_1|T|\mu_2\rangle\langle\mu_2|T|\mu_3\rangle...\langle\mu_{N-1}|T|\mu_N\rangle\langle\mu_N|T|\mu_1\rangle \tag{89}$$

The sums run over all row spins for each of the configuration μ_k, which means ffrom σ_1 to σ_N. We now may carry out the sums over spins, which gives

$$Z(L,N) = \mathrm{Tr}\big[T_L^N\big] \tag{90}$$

The Transfer Matrix defined in Eq. 88 is manifestly symmetric, and therefore diagonalizable which means we have

$$Z(L,N) = \sum_{j=1}^{2^N} \lambda_j^N(L) \tag{91}$$

where λ_j are the eigenvalues of T. The free energy is then

$$F(L,N) = -\frac{1}{\beta}\ln Z = -\frac{1}{\beta}\ln\Big[\lambda_0^N \sum_{j=1}^{2^N}\Big(\frac{\lambda_j}{\lambda_0}\Big)^N\Big] = -\frac{1}{\beta}N\ln\lambda_0 - \frac{1}{\beta}\ln\big[\sum_j (\frac{\lambda_j}{\lambda_0})^N\big] \tag{92}$$

where λ_0 is the largest eigenvalue of T, and given that the matrix entries are positive the largest eigenvalue must non-degenerate. As $|\lambda_j/\lambda_0| < 1$ we find for the free energy per site:

$$\boxed{f = -\frac{1}{\beta}\frac{\ln\lambda_0}{L}} \tag{93}$$

To express correlation functions in terms of the transfer matrix, we introduce a spin operator σ_i acting on our vector space. Then the two-point correlation function reads as:

$$\langle\sigma_i\sigma_{i+l}\rangle = \frac{1}{Z}\sum_{\mu_1,\ldots,\mu_N}\langle\mu_1|T_L^i|\mu_i\rangle\hat{\sigma}_i\langle\mu_i|T_L^l|\mu_{i+l}\rangle\hat{\sigma}_{i+l}\langle\mu_{i+l}|T_L^{N-i-l}|\mu_1\rangle$$

One can show that the corrrelation functions takes a simple form which reads:

$$\boxed{\langle\sigma_i\sigma_{i+l}\rangle = \lim_{N\longrightarrow\infty}\frac{\mathrm{Tr}\Big[T_L^i\hat{\sigma}T_L^l\hat{\sigma}T_L^{N-i-l}\Big]}{\mathrm{Tr}\Big[T_L^N\Big]}} \tag{94}$$

As an example we write down the transfer matrix of the case $N = 2$ of the 2D Ising model:

$$T = \begin{pmatrix} e^{4J} & 1 & 1 & 1 \\ 1 & 1 & e^{-4J} & 1 \\ 1 & e^{-4J} & 1 & 1 \\ 1 & 1 & 1 & e^{4J} \end{pmatrix} \tag{95}$$

One can verify using Eq. 91 that this transfer matrix produces the correct partition function for a 2×2 Ising model.

5 Conformal Mapping to the Cylinder

Now we make some important remarks about conformal mappings and their consequences. It is important to note that we will be using these results in our numerical studies later on (Chap. 6).

5.1 Conformal Transformations of the Energy-Momentum Tensor

We investigate the behaviour of the energy-momentum tensor under conformal transformations. According to the conformal Ward identity Eq. (50) the variation of T under a local conformal transformation is

$$\begin{aligned}\delta_\epsilon T(w) &= \frac{1}{2\pi i}\oint_C dz\epsilon(z)T(z)T(w) \\ &= \frac{1}{12}c\partial_w^3\epsilon(w) + 2T(w)\partial_w\epsilon(z) + \epsilon(w)\partial_w T(w)\end{aligned}$$

Now we will make an Ansatz on how the energy-momentum tensor behaves under conformal transformation:

$$T'(z) = \Big(\frac{dw}{dz}\Big)^2 T(w) + \frac{c}{12}\{w;z\} \tag{96}$$

where $\{w;z\}$ denotes the Schwarzian derivative:

$$\{w;z\} = \frac{d^3w/dz^3}{(dw/dz)} - \frac{3}{2}\Big(\frac{d^2w/dz}{dw/dz}\Big)^2 \tag{97}$$

We will not prove Eq. (96) in detail but verify it on the level of infinitesimal conformal transformations $w(z) = z + \epsilon(z)$. For this transformation the leading order contribution to the Schwarzian derivative reads

$$\{w;z\} = \frac{\partial_z^3\epsilon}{1+\partial_z\epsilon} - \frac{3}{2}\Big(\frac{\partial_z^2\epsilon}{1+\partial_z\epsilon}\Big)^2 \approx \partial_z^3\epsilon \tag{98}$$

The variation of the energy momentum tensor can then be computed as

$$\begin{aligned}\delta_\epsilon T(z) &= T'(z) - T(z) \\ &= (1+\partial_z\epsilon(z))^2\big(T(z) + \epsilon(z)\partial_z T(z)\big) + \frac{c}{12}\partial_z^3\epsilon(z) - T(z) \\ &= \frac{c}{12}\partial_z^3\epsilon(z) + 2T(z)\partial_z\epsilon(z) + \epsilon(z)\partial_z T(z)\end{aligned}$$

which is the same as the variation obtained from the Ward identity. Thus we have verified Eq. (96) at the level of infinitesimal conformal transformations.

5.2 Physical Meaning of c

The appearence of the central charge c is related to a "soft" breaking of conformal symmetry by the introduction of a macroscopic scale into the system. In other words, c describes the way a specific

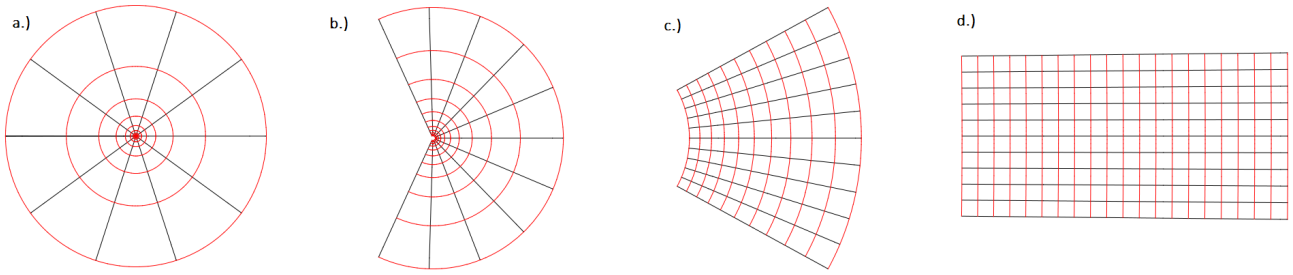

Figure 10: Conformal map Eq. (99) (from a to d). Where b and c are just auxiliary maps to depick how circles of different radius are mapped onto different sections of the cylinder.

system reacts when macroscopic length scales are introduces. To follow up on this statment, we consider a generic conformal field theory living on the whole complex plane, and we map this theory on a strip of width L by the transformation(Fig.10 and Appendix B)

$$w = \frac{L}{2\pi} \ln z \tag{99}$$

The negative part of the real axis corresponds to both edges of the strip, which has therefore the geometry of a cylinder. Under this particular transformation Eq. (96) with $dw/dz = L/(2\pi z)$ and the the Schwarzian derivative is being $1/2z^2$ the energy-momentum tensor $T_{cyl.}(w)$ on the cylinder is related to the corresponding tensor $T_{pl.}(z)$ on the plane by:

$$T_{\text{cyl.}}(w) = \left(\frac{2\pi}{L}\right)^2 \left[T_{\text{pl.}}(z)z^2 - \frac{c}{24}\right] \tag{100}$$

Consequently, the expectation value of T on the cylinder is different from zero

$$\langle T_{\text{cyl.}}(w)\rangle = -\frac{c\pi^2}{6L^2} \tag{101}$$

even if $\langle T_{plane} = 0\rangle$. The central charge is seen to be proportional to the *Casimir energy*, the change in the vacuum energy density brought about by the periodicity condition on the cylinder. The Casimir energy naturally goes to zero as the macroscopic scale L goes to infinity.

5.3 Conformal Invariance on a Cylinder

The relevance of studying models over a finite-size region is manifold. For instance, a lot of information on two-dimensional statistical models or one-dimensional quantum models is derived from computer simulation, which are necessarily limited to systems of finite size L. The properties of the model in the thermodynamic limit ($L \longrightarrow \infty$) are inferred from the finite-size properties. Conformal invariance can in many cases provide the L-dependence of these properties, thus allowing a more precise inference of the thermodynamic limit. This comparison with numerical work may also provide an otherwise unknown correspondence between a model at criticality and a conformal field theory. In quantum systems (e.g. spin chains), the finite size may be in the (imaginary) time direction, which corresponds to finite temperature (Re. [6] Chapter 3). Conformal invariance is then useful in studying the finite-temperature behaviour of a 1D quantum model, which is critical at $T = 0$.

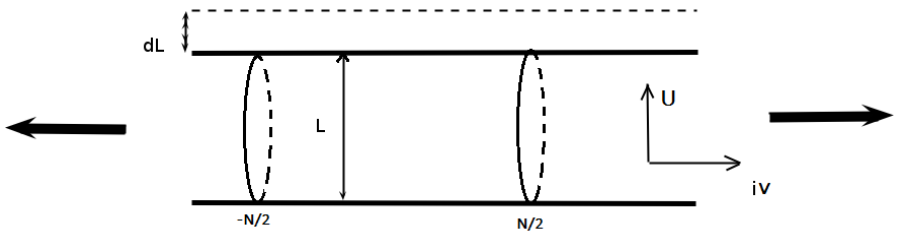

Figure 11: Variation of the width of the cylinder

5.4 Quantities of Interest

Remarks from previous section allow us to relate the central charge to the free energy per unit length of a statistical system defined on a cylinder. In general the partition function of our system on the cylinder geometry reads:

$$Z(L,N) = \int \mathcal{D}\phi e^{-S[\phi]} \quad , \quad S[\phi] = \int_{-N/2}^{N/2} \int_{0}^{L} dl\mathcal{L}(l) \tag{102}$$

and the free energy is defined as an logarithm of the partition function:

$$e^{-F} = \int \mathcal{D}e^{-S[\phi]} \tag{103}$$

We perform an infinitesimal scaling of the circumference: $L \longrightarrow (1+\epsilon)L$ or $\delta L = \epsilon L$. This is realized by applying a coordinate transformation $u \longrightarrow (1+\epsilon)u$, where u is the coordinate running across the cylinder ($w = u + iv$). This variation in terms of Eq. 103 leads to a variational term of the free energy $\delta_\epsilon F$ and in the action $\delta_\epsilon S$. In order to make the expression (before and after variation) have the same form, a condition $\delta_\epsilon F = -\langle \delta_\epsilon S \rangle$ arises. The variation of the action is only defined inside the correlation function[5], and reads:

$$\delta_\epsilon S = \frac{1}{2\pi} \int_{-N/2}^{N/2} \int_{0}^{L} ds^\mu T_{\mu\nu} \partial_\nu \epsilon^\mu = \frac{1}{2\pi} \int_{-N/2}^{N/2} dv \int_{0}^{L} du T_{uu} \tag{104}$$

If S is conformal we know that $T_{uu} = T_{cyl.} + \overline{T}_{cyl.}$. The variation of the energy now reads:

$$\delta_\epsilon F_L = -\epsilon \int_{0}^{L} \langle T_{\text{cyl.}} \rangle du \tag{105}$$

where $\langle T_{\text{cyl.}} \rangle$ is given by Eq. 101. After integration we have

$$\delta_\epsilon F_L = \frac{c\pi}{6L}\epsilon = \frac{c\pi}{6L^2}\delta L \tag{106}$$

[5]Expression of the variation of the action has already been used in Eq. 38

which gives if we regard the variation as a derivative:

$$\frac{dF_L}{dL} = \frac{c\pi}{6L^2} \tag{107}$$

After integrating this equation gives

$$\boxed{F_L = f_0 L - \frac{\pi c}{6L}} \tag{108}$$

This relation is important in the study of finite-size effects of statistical systems and numerical simulations,and we will use it in our study of the 2D Ising model.

Now we will introduce another quantity of interest, two-point correlation function of a primary field ϕ of conformal dimension h. Its form on the plane was fixed by invariance under the global conformal transformation Eq. (25). In order to write the two-point function on the cylinder we need to use the covariance relation Eq. (18) for primary fields, with the mapping Eq. (99). The holomorphic part only reads:

$$\begin{aligned}\langle\phi(w_1)\phi(w_2)\rangle &= \Big(\frac{dw}{dz}\Big)^{-h}_{w=w_1}\Big(\frac{dw}{dz}\Big)^{-h}_{w=w_2}\langle\phi(z_1)\phi(z_2)\rangle \\ &= \Big(\frac{2\pi}{L}\Big)^{2h}\frac{e^{2\pi h(w_1+w_2)/L}}{(z_1-z_2)^{2h}} \\ &= \Big(\frac{2\pi}{L}\Big)^{2h}\big(2\sinh[\pi(w_1-w_2)/L]\big)^{-2h}\end{aligned}$$

The full correlator is the product of the above with its anti-holomorphic counterpart:

$$\langle\phi(w_1,\overline{w}_1)\phi(w_2,\overline{w}_2)\rangle = \Big(\frac{2\pi}{L}\Big)^{2h+2\overline{h}}\big(2\sinh[\pi(w_1-w_2)/L]\big)^{-2h}\big(2\sinh[\pi(\overline{w}_1-\overline{w}_2)/L]\big)^{-2h} \tag{109}$$

For simplicity, we assume that the field ϕ has no spin: $h = \overline{h} = \Delta/2$. Then the above reduces to

$$\Big(\frac{2\pi}{L}\Big)^{\Delta}\Big[4\sinh\frac{\pi w}{L}\sinh\frac{\pi\overline{w}}{L}\Big]^{-\Delta} \tag{110}$$

where $w = w_1 - w_2$ and $\overline{w} = \overline{w}_1 - \overline{w}_1$ are relative coordinates. We express this result in terms of real coordinates u and v, respectively, along and across the cylinder: $w = u + iv$ and $\overline{w} = u - iv$. After using standard identities for hyperbolic functions, we end up with

$$\langle\phi(u_1,v_1)\phi(u_2,v_2)\rangle = \Big(\frac{2\pi}{L}\Big)^{\Delta}\Big[2\cosh\frac{2\pi u}{L} - 2\cos\frac{2\pi v}{L}\Big]^{-\Delta} \tag{111}$$

As expected, the effect of the finite size L disappears if the distance $|u+iv|$ is much smaller then L. When $u \gg L$, then $2\cosh\dfrac{2\pi u}{L} \sim e^{2\pi u/L}$ and the correlator becomes

$$\langle\phi(u_1,v_1)\phi(u_2,v_2)\rangle \sim \Big(\frac{2\pi}{L}\Big)^{\Delta} e^{-\dfrac{2\pi u\Delta}{L}} \qquad \text{for } u \gg L \tag{112}$$

Thus, correlation along the cylinder decay exponentially, with a correlation length

$$\xi = L/2\pi\Delta \tag{113}$$

,proportional to the size L. The appearance of a correlation length in a critical system is here entirely due to the existence of a macroscopic scale L.

When dealing with a quantum chain, the infinite cylinder geometry may correpond either to a finite chain at zero temperature with periodic boundary conditions, or to an infinite chain at a finite temperature $T = 1/L$. In the first case, the correlation length $\xi = L/2\pi\Delta$ in the time direction is the signature of an energy gap between the ground state and the first excited state, a gap introduced by the system's finite size, as routinely observed in numerical simulations. To make this point more explicit, we consider the two-point correlation function $\langle\phi(x,0)\phi(x,\tau)\rangle$, where x nad τ are respectively, the space and imaginary time coordinates. The field operator $\phi(x,\tau)$, given in the Heisenberg picture, can be rewritten in the Schrodinger picture as

$$\phi(x,\tau) = e^{-H\tau}\phi(x,0)e^{H\tau} \tag{114}$$

Now the two-point correlation function reads:

$$\begin{aligned}\langle\phi(x,0)\phi(x,\tau)\rangle &= \langle 0|\phi(x,0)e^{-H\tau}\phi(x,0)e^{H\tau}|0\rangle \\ &= \sum_n \langle 0|\phi(x,0)e^{-H\tau}|n\rangle\langle n|\phi(x,0)e^{H\tau}|0\rangle \\ &= \sum_n e^{-(E_n-E_0)\tau}|\langle 0|\phi(x,0)|n\rangle|^2\end{aligned}$$

Where H is the Hamiltonian, the states $|n\rangle$ are the energy eigenstates(in increasing order of energy), and E_n is the eigenvalue of H associated with $|n\rangle$. In the first line we have performed a time translation to make the two fields simultaneous. In the second line we have inserted a completeness relation for the basis of eigenstates of H. We can rewrite the above equation as the connected two-point correlation function:

$$\langle\phi(x,0)\phi(x,\tau)\rangle - \langle\phi(x,0)\rangle\langle\phi(x,\tau)\rangle = \sum_{n>0} e^{-(E_n-E_0)\tau}|\langle 0|\phi(x,0)|n\rangle|^2 \tag{115}$$

The term that dominates the above sum when τ is large is associated with the first excited state $|1\rangle$, with an energy $\delta E = E_1 - E_0$ above the ground state:

$$\langle\phi(x,0)\phi(x,\tau)\rangle_c \propto e^{-\delta E\tau} \qquad \text{for } \tau \longrightarrow \infty \tag{116}$$

Compairing Eq. (116) with Eq. (112) we conclude that

$$\delta E = \frac{2\pi\Delta}{L} \tag{117}$$

We can expect (through the correspondance between d-dimensional classical statistical theory and $(d-1)$-dimensional quantum theory) that difference of free energies per spin at distict values of L amounts exactly to the the energy gap in the above equation

$$\boxed{f_i(L) - f_0(L) = \frac{2\pi\Delta}{L^2}} \tag{118}$$

This result will help us to extract the conformal dimension Δ for the 2D Ising model through numerical means. One can infer this result from the Transfer Matrix formalisam as well. Starting from Eq. 94 we can compute:

$$\langle\phi_i\phi_{i+l}\rangle \sim const. \cdot \left(\frac{\lambda_i}{\lambda_0}\right)^l = const. \cdot e^{l\ln\frac{\lambda_i}{\lambda_0}} \tag{119}$$

Where the constant term is accually a matrix element. If one compairs Eq. 112 and Eq. 119, relation in Eq. 118 is obtained.

5.5 CFT on the Cylinder and Representation

Now we will give a formal definition of the Hamiltonian in the cylinder geometry:

$$H = \frac{1}{2\pi}\int_0^L T_{uu}(w)du = \frac{1}{2\pi}\int_0^L du\Big(T(w) + \overline{T}(\overline{w})\Big) \tag{120}$$

This definition is the standard way of obtaining the total energy of a field configuration, but adjusted to our paradigm shift of compactifying the theory to a cylinder topology. Now we will express our energy-momentum tensor using Eq. (100) as:

$$\frac{1}{2\pi}\int_0^L duT(w) = \frac{2\pi}{L^2}\int_0^L du\Big[T(z)z^2 - \frac{c}{24}\Big] \tag{121}$$

Now employing Eq. (53)

$$\Rightarrow \frac{2\pi}{L^2}\int_0^L dv\Big[\sum_{n=-\infty}^{\infty} L_n z^{-n} - \frac{c}{24}\Big] == \frac{2\pi}{L}\Big[L_0 - \frac{c}{24}\Big] \tag{122}$$

We obtained L_0 by expressing z as $\exp\frac{2\pi w}{L}$ (with $w = u + iv$)and observing that the integral would produce a non zero value only for $n = 0$. The same derivation works for $\overline{T}(\overline{w})$. Combining the two terms our operator Hamiltonian can be written as being part of the Virasoro algebra

$$\boxed{H = \frac{2\pi}{L}(L_0 + \overline{L}_0) - \frac{\pi c}{6L}} \tag{123}$$

We can express the momentum operator

$$P = \frac{1}{2\pi}\int_0^L T_{uv}du \tag{124}$$

Similarly, we can derive

$$\boxed{P = \frac{2\pi}{L}(L_0 - \overline{L}_0)} \tag{125}$$

These operators commute:

$$\big[H, P\big] = 0 \tag{126}$$

From the explicit expressions for H and P it can be seen that their eigenvectors are in one-to-one correspondance with the eigenvectors of $L_0 + \overline{L}_0$ and $L_0 - \overline{L}_0$.

As it is known from Quantum Mechanics, the generator of time translation is the Hamiltonian which in the present case corresponds to the dilatation operator. This operator finds its geometrical meaning in Fig. 10 where the time translation correspond to the red circles of heigher and higher radius. Similarly, the generator for space translation is the momentum operator corresponding to rotations of points on these circles.

Important thing to note that with our cylinder mapping we were able to represent this time propagation as propagation along the cylinder(circles of different radius are mapped onto different sections of the cylinder). This propagation directly correponds to the propagation induced by the Transfer Matrix, and therefore the eigenvectors of the Transfer Matrix will correpond to eigenvectors L_0 and $\overline{L}_0$:

$$T \sim e^H \sim e^{L_0+\overline{L}_0} \tag{127}$$

5.6 Correlation functions in cylinder geometry

The critical two-point correlation function of a primary operator on the cylnder $w = (N/2\pi)\ln(z)$ is (Eq. (109)):

$$\langle \phi_h(w,\overline{w})\phi_{h'}(w',\overline{w}')\rangle = \Big(\frac{2\pi}{L}\Big)^{2h+2\overline{h}} \big(2\sinh[\pi(w-w')/L]\big)^{-2h} \big(2\sinh[\pi(\overline{w}-\overline{w}')/L]\big)^{-2\overline{h}} \tag{128}$$

where $w = u + iv$, $w' = u' + iv'$ for the holomorphic part. The antiholomoprhic part follows the same. We use the development $(1+x)^\alpha = \sum\limits_{n=0}^{\infty} \binom{\alpha}{n} x^n$ for $u > u'$ we derive the two-point function to be:

$$= \Big(\frac{2\pi}{L}\Big)^{2\Delta} \sum_{N=0}^{\infty}\sum_{N'=0}^{\infty} a_N a_{N'} \exp\big[-\frac{2\pi}{L}(u-u')(h+\overline{h}+N+N')\big] \exp\big[-\frac{2\pi i}{L}(v-v')(h-\overline{h}+N-N')\big] \tag{129}$$

where $\Delta = h + \overline{h}$, $s = h - \overline{h}$ are the scaling dimension and the spin of the fields respectively and

$$a_N = \frac{\Gamma(2h+N)}{\Gamma(2h)N!} \quad , \quad a_{\overline{N}} = \frac{\Gamma(2\overline{h}+N')}{\Gamma(2\overline{h})N!} \tag{130}$$

this follows from the binomial theory and the identity

$$\binom{-2h}{k} = (-1)^k \frac{\Gamma(2h+k)}{\Gamma(2h)k!} \tag{131}$$

On the other hand, the correlation function in the strip may be evaluated using transfer matrix techniques. In that case the scaling "operators" $\phi(u,v)$ become true operators $\hat{\phi}(u)$ acting on the same Hilbert space as does the transfer matrix. The time and space translation of these operators in the Heisenberg picture are:

$$\phi(u,v) = e^{Hv}\phi(u,0)e^{-Hv} \quad , \quad \phi(0,u) = e^{-iPu}\phi(0,0)e^{iPu} \tag{132}$$

The correlation function now can be written down as

$$\langle \phi(w,\overline{w})\phi(w',\overline{w}'))\rangle = \sum_{n,k} \langle 0|\phi(0,0)|n,k\rangle e^{-(E_n-E_0)(v-v')+ik(u-u')} \langle n,k|\phi(0,0)|0\rangle \tag{133}$$

where we used the completeness relation and where $|n,k\rangle$ is a complete set of eigenstates of $\hat{H}$ of energy E_n and momentum k. Compairing with Eq. (129), we see that to each primary operator of

dimension Δ and spin s corresponds an infinite number of eigenstates of $\hat{H}$, labelled by $(N, \overline{N})$, with energy and momentum being:

$$E_n = E_0 + \frac{2\pi}{L}(\Delta + N + N') \quad , \quad p_N = \frac{2\pi}{L}(s + N - N') \tag{134}$$

where $E_0 = -\pi c/6L$. The lowest such state must be non-degenerate, and we denote it by $|\phi\rangle$. From Eq. (129) we see that

$$\langle 0|\hat{\phi}(v)|\phi\rangle = \Big(\frac{2\pi}{L}\Big)^{\Delta} \tag{135}$$

One can verify that the constant term in Eq. 119 is accually the matrix element in Eq. 135. Only matrix elements involving the lowest states $|\phi\rangle$ have this simple form as in Eq. (135). To obtain further matrix elements we need to consider the three-point correlation function on the strip:

$$\langle \phi_{h_1}(w_1,\overline{w}_1)\phi_{h_2}(w_2,\overline{w}_2)\phi_{h_3}(w_3,\overline{w}_3)\rangle = \Big(\frac{dw}{dz}\Big)^{-h_1}\Big|_{w=w_1}\Big(\frac{dw}{dz}\Big)^{-h_2}\Big|_{w=w_2}\Big(\frac{dw}{dz}\Big)^{-h_3}\Big|_{w=w_3}\times \\ \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}} \times \text{anti-holomorphic part}$$

where $z_{i,j} = z_i - z_j$. C_{123} is the operator product expansion coefficient of ϕ_3 in the expansion of ϕ_1 and ϕ_2. This quantity is usually referred to as *structure constant*. In the strip geometry one finds when developed for $u_1 \gg u_2 \gg u_3$:

$$\begin{aligned}
&\langle \phi_{h_1}(w_1,\overline{w}_1)\phi_{h_2}(w_2,\overline{w}_2)\phi_{h_3}(w_3,\overline{w}_3)\rangle = \\
&=\Big(\frac{2\pi}{L}\Big)^{\Delta_1+\Delta_2+\Delta_3}\exp\Big[\frac{2\pi}{L}(u_3-u_2)\Delta_3\Big]\exp\Big[\frac{2\pi}{L}(u_2-u_1)\Delta_1\Big]\exp\Big[\frac{2\pi}{L}i(v_2-v_3)s_3\Big]\exp\Big[\frac{2\pi}{L}i(v_1-v_2)s_1\Big] \\
&\times C_{123}\sum_{N,N',N''=0}^{\infty} a_N a_{N'} a_{N''}\exp\Big[\frac{2\pi}{L}(u_2-u_1)(\Delta_3-\Delta_2-\Delta_1)N\Big]\exp\Big[\frac{2\pi}{L}(u_3-u_2)(\Delta_1-\Delta_2-\Delta_3)N'\Big] \\
&\exp\Big[\frac{2\pi}{L}(u_3-u_1)(\Delta_2-\Delta_1-\Delta_3)N''\Big]\exp\Big[\frac{2\pi}{L}i(v_2-v_1)(s_3-s_1-s_2)N\Big] \\
&\exp\Big[\frac{2\pi}{L}i(v_3-v_2)(s_1-s_2-s_3)N'\Big]\exp\Big[\frac{2\pi}{L}i(v_3-v_1)(s_2-s_1-s_3)N''\Big]
\end{aligned}$$

where the development coefficants are:

$$a_N = \frac{\Gamma(h_3-h_2-h_1+N)}{\Gamma(h_3-h_2-h_1)N!} \quad , \quad a_{N'} = \frac{\Gamma(h_1-h_2-h_3+N')}{\Gamma(h_1-h_2-h_3)N'!} \quad , \quad a_N = \frac{\Gamma(h_2-h_1-h_3+N'')}{\Gamma(h_2-h_1-h_3)N''!} \quad , \tag{136}$$

Further on we will be interested just in the lowest state for which $N, N', N'' = 0$, which simplifies our three-point function expression. The same correlation function evaluated in the transfer matrix formalism at its lowest state is:

$$\langle 0|\hat{\phi}_1(v_1)|\phi_1\rangle\exp\Big[-\frac{2\pi}{L}\Delta_1(u_2-u_1)\Big]\langle\phi_1|\hat{\phi}_2(v_2)|\phi_3\rangle\exp\Big[-\frac{2\pi}{L}\Delta_3(u_3-u_2)\Big]\langle\phi_3|\hat{\phi}_3(v_3)|0\rangle \tag{137}$$

Compairing these expressions, and using Eq. (135), one finds:

$$\langle\phi_1|\hat{\phi}_2(v)|\phi_3\rangle = \Big(\frac{2\pi}{L}\Big)^{\Delta_2}\exp\Big[\frac{2\pi}{L}i(s_1-s_3)v\Big]C_{123} \tag{138}$$

Thus the universal operator product expansion coefficients C_{123} are measurable in terms of matrix elements of operators between low-lying states. In practice, the operators $\hat{\phi}_2$ will not be normalized (and the spin of the field will be zero), in which case the structure constants may be obtained from

$$\boxed{C_{123} = \frac{\langle \phi_1 | \hat{\phi}_2(v) | \phi_3 \rangle}{\langle \phi_2 | \hat{\phi}_2(\sigma) | 0 \rangle}} \tag{139}$$

One can identify these operators as being the spin operators inserted in the trace of transfer matrix multiplication for the two-point correlation function as in Eq. **??**. Thus if we can find explicit expression for these operators we can simply act with them on our low-lying states (eigenvectors of the transfer matrix) and compute the structure constant numerically. Pure analytical approach to calculating the structure constant involves using the Coulomb gas approximation, which goes out of the scope of this thesis. [6]

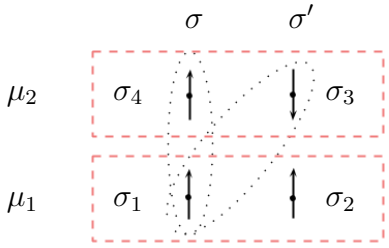

Figure 12: Possible two-point correlation between spin configurations μ_1 and μ_2

As an example we write down the possible operators for the system where $L = 2$, between $\sigma_1\sigma_4$ and $\sigma_1\sigma_3$ (keeping in mind $\sigma_1\sigma_2$ and $\sigma_1\sigma_4$ are the same), depicked in Fig. 12 as

$$\hat{\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \quad \hat{\sigma}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{140}$$

Acting with these operators on the spin states $|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle$ with $|\uparrow\uparrow\rangle$ as the ground state we have:

$$\begin{aligned} \hat{\sigma}|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle \quad &, \quad \hat{\sigma}'|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle \\ \hat{\sigma}|\downarrow\uparrow\rangle = -|\downarrow\uparrow\rangle \quad &, \quad \hat{\sigma}'|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle \\ \hat{\sigma}|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle \quad &, \quad \hat{\sigma}'|\uparrow\downarrow\rangle = -|\uparrow\downarrow\rangle \\ \hat{\sigma}|\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle \quad &, \quad \hat{\sigma}'|\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle \end{aligned}$$

These operators when replaced inside the trace in Eq. (94) produce the correct value for the two-point function evaluated as an ordinary average $\langle \sigma\sigma' \rangle = \frac{1}{Z} \sum_{\{\sigma_N\}} \sigma\sigma' \exp\big(-\beta H(\{\sigma_N\})\big)$.

[6] An introduction to the Coulomb gas formalism one can find in Ref.[9] , [6], [5]

This makes the identification $\phi_\Delta \longrightarrow \hat{\phi}_2 \longrightarrow \hat{\sigma}$ as *conformal field* $\longrightarrow$ *transfer matrix operator* $\longrightarrow$ *spin operator* consistent and useful in our calculations. The spin operators can be written down for any size L.

Graphical representation of the inset of the operators in our strip geometry is given in Fig. 13, where transfer matrices T evolve or build our state space, spin operators σ act on the spins between configurations.

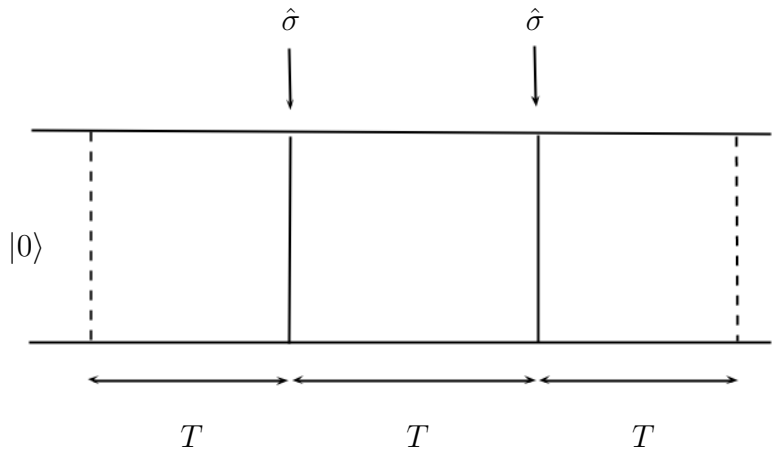

Figure 13: Graphical representation of inset of spin operators σ

The structure constants that we have for the Ising case are:

$$C_{\sigma,\sigma,\epsilon} = \frac{\langle\sigma|\hat{\sigma}|\epsilon\rangle}{\langle\sigma|\hat{\sigma}|0\rangle} \quad , \quad C_{\epsilon,\epsilon,\epsilon} = \frac{\langle\epsilon|\hat{\epsilon}|\epsilon\rangle}{\langle\epsilon|\hat{\epsilon}|0\rangle} \tag{141}$$

These quantities now are easy to compute, given that the states $|\epsilon\rangle$, $|\sigma\rangle$, are the eigenstates of the transfer matrix.

6 Numerical results

Given previous results (Eq. 108, 118) we would like to calculate the free energy of our statistical model. We have seen that the calculation of this thermodynamic quantity boils down to determining the highest eigenvalues, and we implement numerical techiques to do this. A suitable numerical scheme which we use in our study is the Lanczos algoritm (Appendix C). The algorithm uses the Gram - Schmidt procedure in order to build an orthonormal basis.

We preformed our calculation in Python programming enviroment (Ref. [19]) with the help of a Lanczos diaginalization library (Ref. [20]).

6.1 Central charge

We use Eq. 108 in order to make finite size estimates for the central charge. This equation provides a suitable fit that reads:

$$f_0(L) = f_0(\infty) - \frac{\pi c(L)}{6L^2} + \frac{A}{L^4} + \cdots \tag{142}$$

,where A is a fitting constant. Estimates for the central charge can be now extracted from three-point fits of $f_0(L)$, $f_0(L+1)$ and $f_0(L+2)$. This way we obtain Fig. 14. In the plots we have used the notation L^* which is the fit over lattice sizes $L, L+1, L+2$.

We observe an interesting behaviour as we change the inverse temperature given as β. As we increase β (from 0.2) our c value moves continuously to the value $c \sim 0.5$ (at around $\beta_c = 0.44$). At this point the the system is critical, and the correlation length is infinite(because it is a finite system, the correlation length is actually as big as the size of the system). This means that that the central charge acquare a constant value according to $c(L) \sim e^{-\frac{L}{\xi}}$. After the critical point is reached, further increase of the inverse temperature will make the value of c move to zero (Fig. 14 plot (c)).

Our found value $\beta = 0.44$ is close to the exact analytical solution found in the literature, of the 2D Ising model

$$\beta_c = -\frac{1}{2}\ln\left(\sqrt{2}-1\right) = 0.440687 \tag{143}$$

Fitting errors for the central charge are between $0.1\% - 0.5\%$.

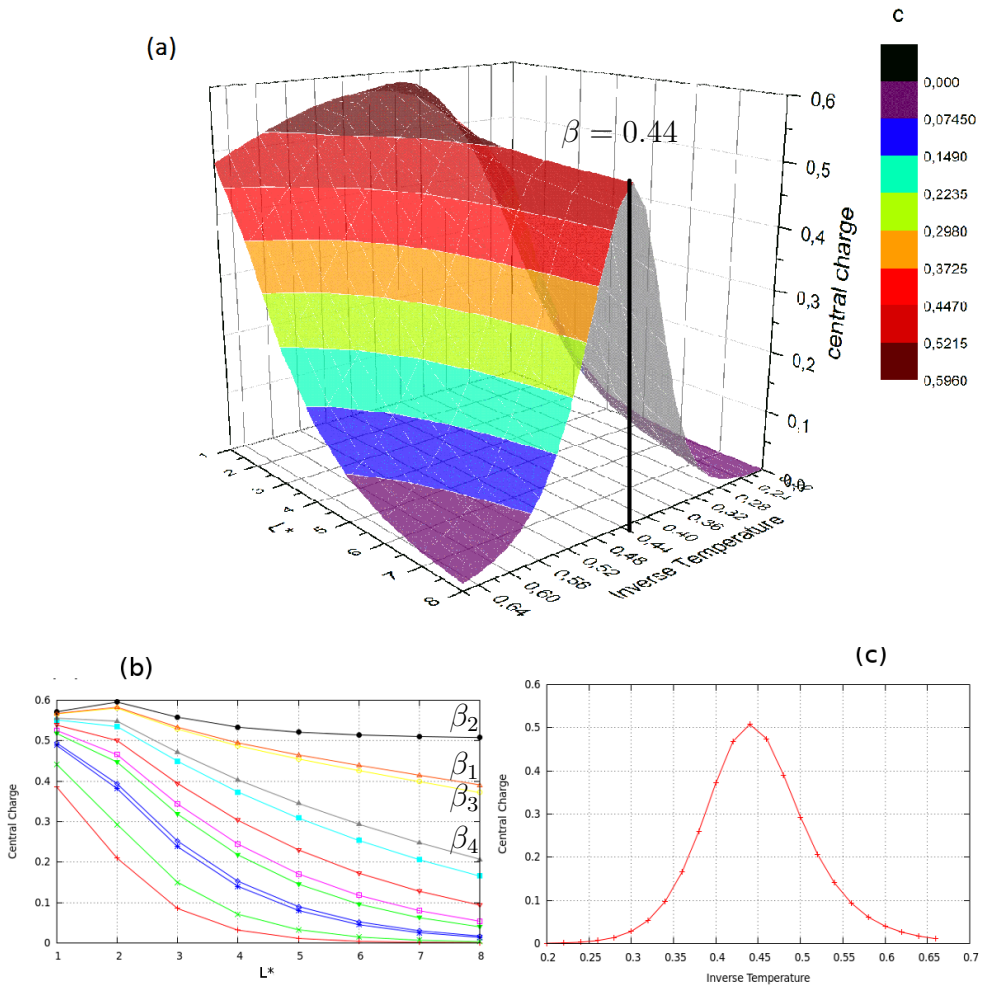

Figure 14: (a) Plot of the central charge over different inverse temperatures β, and L^* which are fits over $L, L+1, L+2$ (b) Plot of the central charge over L^* for different inverse temperatures in range $0.2 < \beta < 0.6$ ($\beta_1 = 0.42$, $\beta_2 = 0.44$, $\beta_3 = 0.48$ and $\beta_4 = 0.52$), (c) Plot of $L^* = 8$ (the largest lattices possible) of the central charge for different β. Lattice sizes considered are $4 < L < 13$.

6.2 Conformal dimension and Verma module structure

Now that we have succesfully identified the critical point we can procede to examine the behaviour of the model at the critical point. At the critical point the system is conformal invariant. The system will be characterised by a finite set of conformal towes, each with certain conformal dimension Δ. In order to calculate these quantities we procede to evaluate the biggest eigenvalues of our transfer matrices, at the critical point. In Fig. 15 we plot the difference between the logarithm of the largest eigenvalues and other eigenvalues. In our diagonalization procedure the eigenvalues are ordered as $\lambda_0 > \lambda_1 > \lambda_2 > \cdots$. To extract the conformal dimension we turn to the Eq. 118, and use the one-point fit for $f_k(L)$, where the index $k = 0, 1, 2, 3...$ indicates which of the highest eigenvalues we are considering

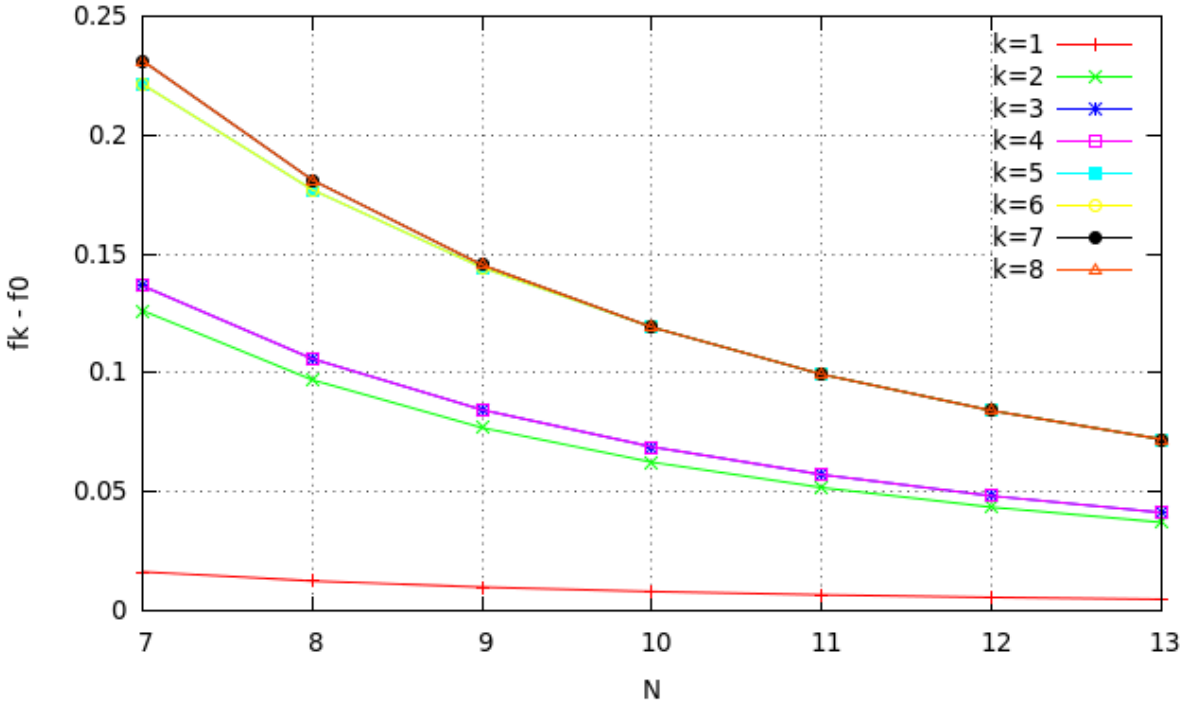

Figure 15: Change of the difference in logarithm of eigenvalues over different lattice sizes N at the critical point

$$f_k(L) - f_0(L) = \frac{2\pi\Delta_k}{L^2} + \frac{B}{L^4} + ... \tag{144}$$

,where B is a fitting constant. The values we obtain are listed in Table 3. Values are listed with their fitting errors. From the list we observe that the fitting errors increase with every eigenvalue, this is because of the Lanczos algorithm which inherently loses in precision as we calculate first, second, third to the highest eigenvalue of a matrix.

Each of the values Δ_k correspond to one of the primary fields. We also observe a degeneration in the values, and this is due to the underlying translational symmetry (space translation) of our model. This means that we need to write down an momentum operator that satisfies:

$$\hat{P}|\psi_k\rangle = e^{ip}|\psi_k\rangle \tag{145}$$

where $|\psi_k\rangle$ is the eigenvector of the eigenvalue λ_k, and p is the momentum quantum number. This operator must commute with the Hamiltonian:

$$\left[\hat{H}, \hat{P}\right] = 0 \tag{146}$$

From path integral formalism, and consideration from previous chapter, we know that the relation between the Transfer Matrix and the Hamiltonian reads as $T \sim e^{-H}$, therefore verifing the commutator

$$\left[\hat{T}, \hat{P}\right] = 0 \tag{147}$$

will be enough to confirm the translational invariance. In order to calculate the momentum of degenerate states we need to diagonalize the matrix (the case if we have two degenerate states):

k	Δ	fitting error for a	p	Primary Field
1	0.124954	$\pm 5.091 \cdot 10^{-6}$ (0.004074 %)	0	σ
2	0.999824	$\pm 2.67 \cdot 10^{-5}$(0.00267%)	0	ϵ
3	1.12256	$\pm$0.0003538(0.03152%)	± 1	σ
4	1.12256	$\pm$0.007506(0.03152%)		
5	2.04204	$\pm$0.007506(0.3676%)	± 1	ϵ
6	2.04204	$\pm$0.007506(0.3676%)		
7	1.98511	$\pm$0.001975(0.09947%)	± 2	I
8	1.98511	$\pm$0.001975(0.09947%)		
9	2.01972	$\pm$0.01458(0.7218%)	± 2	σ
10	2.01972	$\pm$0.01458(0.7218%)		
11	2.12011	$\pm$0.0007(0.03302%)	0	σ
12	2.91254	$\pm$0.02127(0.7301%)	± 3	I
13	2.91254	$\pm$0.02127(0.7301%)		
14	2.87815	$\pm$0.0406(1.411%)	± 2	ϵ
15	2.88751	$\pm$0.01238(0.4287%)		
16	2.92029	$\pm$0.04243(1.453%)	± 3	σ
17	2.97181	$\pm$0.1223(4.114%)		
18	3.07037	$\pm$0.1249(4.068%)	0	ϵ

Table 3: Table of values obtained at the critical point using the fit Eq. 144

$$M = \begin{pmatrix} \langle\psi_k|\hat{P}|\psi_k\rangle & \langle\psi_k|\hat{P}|\psi_{k+1}\rangle \\ \langle\psi_{k+1}|\hat{P}|\psi_k\rangle & \langle\psi_{k+1}|\hat{P}|\psi_{k+1}\rangle \end{pmatrix} \tag{148}$$

Diagonalization of M will provide us the eigenvalues as $e^{\frac{2\pi i}{N}p_k}$. As an illustration of a momentum operator for $N = 2$ we have:

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{149}$$

which commutes with T given in Eq. 95. Acting with this operators on the spin states $|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle$ produces

$$\begin{aligned} \hat{P}|\uparrow\uparrow\rangle &= |\uparrow\uparrow\rangle \\ \hat{P}|\downarrow\uparrow\rangle &= |\uparrow\downarrow\rangle \\ \hat{P}|\uparrow\downarrow\rangle &= |\downarrow\uparrow\rangle \\ \hat{P}|\downarrow\downarrow\rangle &= |\downarrow\downarrow\rangle \end{aligned}$$

Furthermore all these results admit a Verma module structure consisting out of our three primary fields $\phi_{(1,1)}, \phi_{(\frac{1}{2},\frac{1}{2})}\phi_{(\frac{1}{16},\frac{1}{16})}$ and are depicked in Fig. 16, 17, 18.

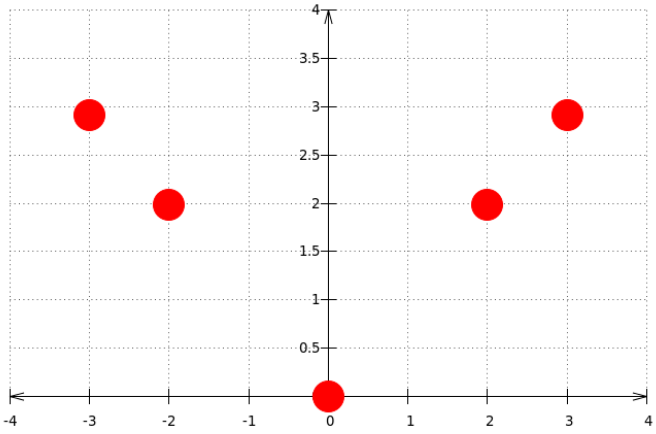

Figure 16: Conformal tower for the identity field I

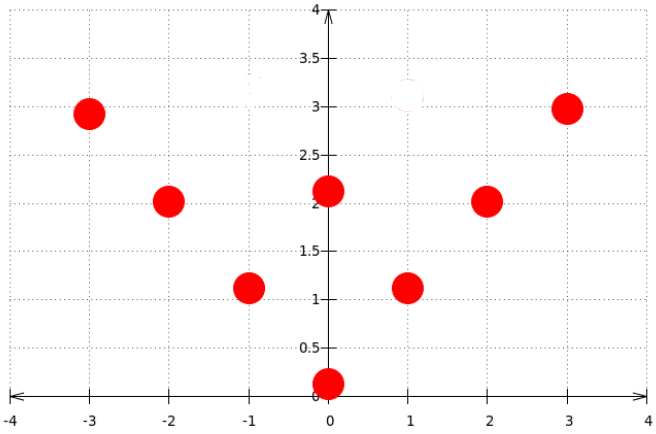

Figure 17: Conformal tower for the identity field σ

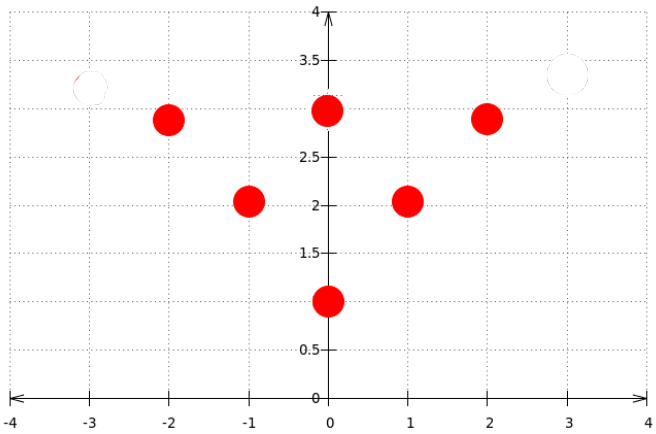

Figure 18: Conformal tower for the identity field ϵ

L	$C_{\sigma,\sigma,\epsilon}$	$C_{\epsilon,\epsilon,\epsilon}$
4	0.434993	0
5	0.455907	0
6	0.468349	0
7	0.476264	0
8	0.48158	0
9	0.485308	0
10	0.488019	0
11	0.490048	0
12	0.491605	0
13	0.492825	0

Table 4: Table of values for the structure constants.

6.3 Critical exponents

From previous considerations Ch. 3.2.1 we verify the universal critical exponent values to be $\eta = 1/4$ and $\nu = 1$ given that we have calculated $\Delta^\sigma = 0.124987 \sim 1/8$ and $\Delta^\epsilon = 0.999824 \sim 1$

6.4 Structure constant

We procede to calculate the structure constant, and the results we obrain are listed in Table 4.

The results in the table, can be fitted in order to extract the thermodynamic limit. The fitting function one uses:

$$f(L) = \frac{L}{1.885(L+1)} \tag{150}$$

With this function in the limit $L \longrightarrow \infty$, one finds for the structure constant $C_{\sigma,\sigma,\epsilon} = 0.54 \pm 0.01$. These result agree with the analytic calculation in Ref. [5] (using Coulomb gas approach), and numerical Monte Carlo calculation in Ref. [22].

7 Numerical code

All Python scripts that were used in this work can be found https://github.com/dzovan137/CFT

8 Acknowledgments

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In the end I would like to thank Ill de France regional council, for their financial support during my stay in Paris.

Appendices

A Möbius Transformation

A Möbius transformation, or a bilinear transformation, is a rational fuction $T : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d} \tag{A-1}$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$.

Geometrically, a Möbius transformation can be obtained by preforming a sterographic projection from the plane to the Riemann sphere, rotating and moving the sphere to a new location and orientation in space, then reprojecting on the plane again. Hence the name *projective conformal transformation*. These transformation preserve angles, map every straight line to a line or circle, and map every circle to a line or circle.

In the context of previously mentioned conformal transformation (translation, rotation, dilation, SCT in Table 1) in the complex plane we have

$$\begin{aligned}
&z \to z' = z + A \quad , \quad \text{translation} \\
&z \to z' = Bz \quad , \quad \text{rotation by angle } B = e^{\theta}, \theta \in \mathbb{R} \\
&z \to z' = Bz \quad , \quad \text{dilation by} B = \alpha \in \mathbb{R} \\
&z \to z' = \frac{z}{1 + Dz} \quad , \quad \text{SCT}
\end{aligned}$$

Where for SCT we had to define $D = D_1 - iD_2$ and compute as:

$$z' = \frac{z + \overline{D}z\overline{z}}{1 + \overline{D}\overline{z} + az + D\overline{D}z\overline{z}} = \frac{z(1 + \overline{D}\overline{z})}{(1 + Dz)(1 + \overline{D}\overline{z})} = \frac{z}{1 + Dz} \tag{A-2}$$

By combining these transofrmation, one obtaines the *global conformal transformation* or *Möbius transformation* in two dimension:

$$z' = \frac{Bz + A}{1 + D(Bz + A)} = \frac{Bz + A}{(1 + BD) + BDz} \tag{A-3}$$

where the general form is the Möbius transformation Eq. A-1

Important virtues of Möbius transformations is that they map circles to circles and they are the only conformal maps of the sphere to itself. These are each global characterizations of Mobius transformations, while an infinitesimal characterization is the vanishing of the Schwarzian. To see this, if we take the derivative of Eq. (A-1)

$$f'(z) = \frac{1}{(cz + d)^2} \tag{A-4}$$

so

$$u(z) = f'(z)^{-1/2} = cz + d \tag{A-5}$$

Hence

$$\begin{aligned} 0 = u'' &= \frac{3}{4}(f')^{-5/2}(f'')^2 - \frac{1}{2}(f')^{-3/2}f''' \\ &= -\frac{1}{2}(f')^{-1/2}\Big(- \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2 + \frac{f'''}{f'}\Big) \\ &= -\frac{1}{2}f'(z)^{-1/2}\{f, z\} \end{aligned}$$

which delivers the Schwarzian derivative **??** to us and proves that it vanishes identically exactly for the Möbius transformations.

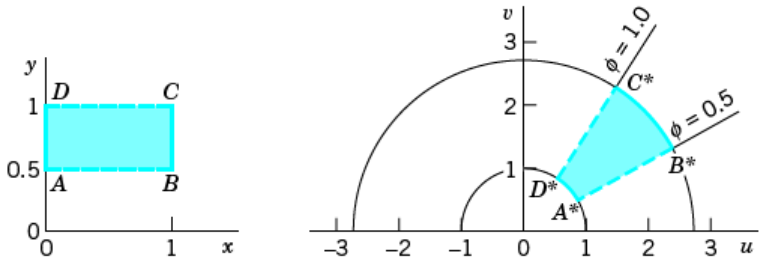

Figure 19: Mapping by $w = e^z$

B Mapping and Transformations

Here we will introduce a few of the techniques that can be used to transfrom a region of the complex plane into another different region of the complex plane.

B.1 Conformal map $w = e^z$

A consequence of the Euler formula

$$e^{iy} = \cos y + i \sin y \tag{B-1}$$

is

$$|e^{iy}| = |\cos y + i \sin y| = 1 \tag{B-2}$$

That is, for pure imaginary exponents, the exponential function has the modulus equal to 1. We see that for the exponential map $w = e^z$ where $z = x + iy$

$$|e^z| = e^x \qquad \arg e^z = y \pm 2n\pi \tag{B-3}$$

Hence e^z maps a vertical straight line $x = x_0 = const.$ onto the circle $|w| = e^{x_0}$ and a horizontal straight line $y = y_0 = const.$ onto the ray $\arg w = y_0$. The rectangle in Fig. 19 is mapped onto a region bounded by circles and rays as shown.

The fundamental region $-\pi < Argz \leqq \pi$ of e^z in the z - plane is mapped bijectively and conformally onto the entire w - plane without the origin $w = 0$ (because $e^z = 0$ for no z). Fig. 20 shows that the upper half $0 < y \leqq \pi$ of the fundamental region is mapped onto the upper half-plane $0 < \arg w \leqq \pi$, the left half being mapped inside the unit disk $|w| \leqq 1$ and the right half outside.

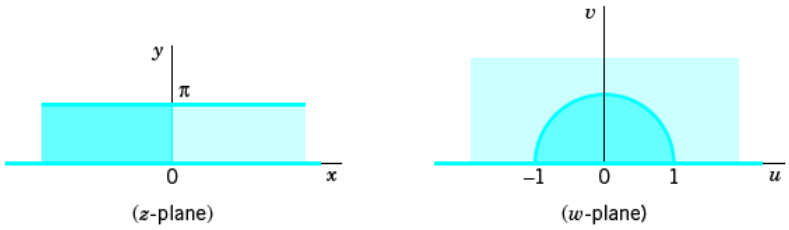

Figure 20: Mapping by $w = e^z$

B.2 Inverse mapping. Mapping $w = \mathrm{Ln}z$

The mapping by the inverse $w = f^{-1}(w)$ of $w = f(z)$ is obtained by interchanging the roles of the z-plane and the w - plane in the mapping by $w = f(z)$.

Now the principal value $w = f(z) = \mathrm{Ln}z$ of the natural logarithm has the inverse $z = f^{-1}(w) = e^w$. From previouse subsection we know that $f^{-1}(w) = e^w$ maps the fundamental region of the exponential function onto the z-plane without $z = 0$. Hence $w = f(z) = \mathrm{Ln}$ maps the z-plane without the origin and cut along the negative real axis (where $\theta = \mathrm{Im}\mathrm{Ln}z$ jumps by 2π) conformally onto the horizontal strip $-\pi < v \leqq \pi$ of the w-plane, where $w = u + iv$.

Since the mapping $w = \mathrm{Ln}w + 2\pi i$ differs from $w = \mathrm{Ln}z$ by the translation $2\pi i$(vertically upward), this function maps the z - plane (cut as before and 0 omitted) onto the strip $\pi < v \leqq 3\pi$. Similirarly for each of the infinitely many mappings $w = \ln z = \mathrm{Ln}z \pm 2\pi ni (n = 0, 1, 2...)$. The corresponding horizontal strips of width 2π (images of the z-plane under these mappings) together cover the whole w - plane without overlapping.

C Lanczos algorithm

The Lanczos algorithm is an iterative algorithm, which evaluates the eigenvalues and eigenvectors of a square matrix or singular value decomposition of a rectangular matrix. The algorithm is optimal if we are interested in the highest eigenvalues of a matrix. The idea is to start with a first eigenvector (state) $|0\rangle$ which is chosen at random and which we can apply to the transfer matrix T (or in the quantum mechanical case the Hamiltonian H). The we construct an orthonormal vector $|1\rangle$ to it by:

$$|1\rangle = \frac{1}{b_1}\Big(T|0\rangle - a_0|0\rangle\Big) \tag{C-1}$$

where

$$a_0 = \langle 0|T|0\rangle \quad \text{and} \quad b_1 = ||T|0\rangle - a_0|0\rangle||. \tag{C-2}$$

One constructs the new state as

$$|2\rangle = \frac{1}{b_2}\Big(T|1\rangle - b_1|0\rangle - a_1|1\rangle\Big) \tag{C-3}$$

where

$$a_1 = \langle 1|T|1\rangle \quad \text{and} \quad b_2 = ||T|1\rangle - b_1|0\rangle - a_1|1\rangle|| \tag{C-4}$$

Now states $|0\rangle$ and $|1\rangle$ are both orthonormal to $|2\rangle$. If we iterate on this on this construction we can build up a orthonormal sub-spaces. In general case, state $|n+1\rangle$ is built from $|n\rangle$ and $|n-1\rangle$ and we use:

$$|n+1\rangle = \frac{1}{b_{n+1}}\Big(T|n\rangle - b_n|n-1\rangle - a_n|n\rangle\Big) \tag{C-5}$$

where

$$a_n = \langle n|T|n\rangle \quad \text{and} \quad b_{n+1} = ||T|n\rangle - b_n|n-1\rangle - a_n|n\rangle||. \tag{C-6}$$

According to the generalised formula we are dealing with a sub-space transfer matrix constructed from states $|0\rangle, |1|, |2\rangle, ...|n\rangle$ which has a triagonal form:

$$T_n = \begin{pmatrix} a_0 & b_1 & 0 & ... \\ b_1 & a_1 & b_2 & ... \\ 0 & b_2 & ... & \\ ... & & & a_n \end{pmatrix} \tag{C-7}$$

After the matrix is evaluated, one can find its eigenvalues and corresponding eigenvectors(at each step of the iteration). It can be proved that the these eigenvalues are approximate eigenvalues of the original matrix T.

8.1 List of Figures

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9 References

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