Kink W state magic derivation

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1 Introduction

While performing research for Ref. (ODAVIĆ et al., 2022) I did a small calculation that did not make it to the final version of the manuscript. Here I present that calculation. This evaluation does not rely upon the stabilizer circuit mapping between the W and the kink W state. It verifies the result we obtain using the W state exclusively in the paper.

2 Introduction

To quantify the amount of non-stabilizerness (or "magic") for a generic state defined on a one-dimensional system made of N qubits, it is possible to use the Stabilizer 2-Rényi Entropy (SRE) (LEONE; OLIVIERO; HAMMA, 2022) that is defined as

$$\mathcal{M}_2(|\psi\rangle) = -\log_2\left(\frac{1}{2^N}\sum_P \langle\psi| P |\psi\rangle^4\right), \tag{1}$$

where the sum on the right-hand side runs over all possible Pauli strings $P = \bigotimes_{j=1}^N P_j$ for $P_j \in \{\sigma_j^0, \sigma_k^x, \sigma_j^y, \sigma_j^z\}$ where σ_j^0 stands for the identity operator on the j-th qubit.

3 Inspection approach for the kink W-state

The initial starting point of the whole project was looking at the superposition of kink states ($h \to 0^+$ limit of the transverse-field Ising chain with frustrated boundary conditions)

$$|W_{\mathrm{K}}^{(0)}\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\prod_{j=1}^{\lfloor N/2\rfloor}\sigma_{2j-1+i}^x|0\rangle^{\otimes N}, \tag{2}$$

where with superscipt we denote the state parity. For example for $N = 5$ the state takes the form $|W_{\mathrm{K}}^{(0)}\rangle = \frac{1}{\sqrt{5}}(|00101\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle)$. The following consideration are equivalent for the opposite parity state. We find that the contributions of all possible Pauli strings to the state in Eq. 2 are encapsulated with the following expression

$$\mathcal{M}_\alpha(|W_{\mathrm{K}}^{(0)}\rangle) = \frac{1}{1-\alpha}\log_2\left(2^N\left(\sum_{k=1}^{N} c_k I_k^\alpha(N)\right)\right), \tag{3}$$

where the I_k contribution takes the form

$$I_k(N) = \frac{1}{2^N}\frac{1}{N^2}k^2. \tag{4}$$

The coefficients c_k count the number of occurrences of a particular type of contribution. For k even, with exception of $k = 2$, the number of occurrences is zero. In particular, in Table 1 we

N	4^N	zeros	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
3	64	44	6	12	2								
5	1024	832	20	160	10		2						
7	16384	14912	70	1344	42		14		2				
9	262144	252416	252	9216	168		72		18		2		
11	4194300	4135936	924	56320	660		330		110		22		2

Table 1: Counting using numerics of different types of contribution to the magic for states with finite N. Where there is no number, a zero value is assumed.

tabulate the number of different contributions for states with finite and odd N. We observe a pattern which allows us to express the SRE for $\alpha = 2$ in the following form

$$\mathcal{M}_2(|W_{\mathrm{K}}^{(0)}\rangle) = -\log\left(2\sum_{k=0}^{(N-1)/2}\binom{N}{k}\big(I_{N-2k}(N)\big)^2 + 2^{N-2}N(N-1)I_2^2(N)\right) - N\log 2 \tag{5}$$

Which after plugging Eq. 4 and after expanding yields the following expression

$$\begin{aligned}\mathcal{M}_2(|W_{\mathrm{K}}^{(0)}\rangle) = -\log\Bigg(\frac{2}{2^{2N}N^4}\Bigg(N^4\sum_{k=0}^{(N-1)/2}\binom{N}{k} - 8N^3\sum_{k=0}^{(N-1)/2}k\binom{N}{k} + 24N^2\sum_{k=0}^{(N-1)/2}k^2\binom{N}{k}\\ - 32N\sum_{k=0}^{(N-1)/2}k^3\binom{N}{k} + 16\sum_{k=0}^{(N-1)/2}k^4\binom{N}{k}\Bigg) + \frac{2^{2-N}}{N^3}(N-1)\Bigg) - N\log 2.\end{aligned} \tag{6}$$

To further simplify the expression we use the following identities for odd N

$$\sum_{k=0}^{(N-1)/2}\binom{N}{k} = 2^{N-1}, \tag{7}$$

$$\sum_{k=0}^{(N-1)/2}k\binom{N}{k} = 2^{N-2}N - \frac{1}{2}\left(\frac{N+1}{2}\right)\binom{N}{\frac{N-1}{2}}, \tag{8}$$

$$\sum_{k=0}^{(N-1)/2}k^2\binom{N}{k} = \frac{1}{8}N(N+1)\left(2^N - 2\binom{N}{\frac{N-1}{2}}\right), \tag{9}$$

$$\sum_{k=0}^{(N-1)/2}k^3\binom{N}{k} = \frac{1}{16}\left(2^N N^2(N+3) - (N+1)^2(3N-1)\binom{N}{\frac{N-1}{2}}\right), \tag{10}$$

$$\sum_{k=0}^{(N-1)/2}k^4\binom{N}{k} = \frac{1}{32}N(N+1)\left(2^N(N^2+5N-2) - 4(N^2+2N-1)\binom{N}{\frac{N-1}{2}}\right). \tag{11}$$

The first of these identities is easy to prove by using the known identity $\sum_{k=0}^{N}\binom{N}{k} = 2^N$ and realizing that the sum we are looking for is just half of the known result. On contrary, the remaining ones require some extra effort. In particular, for the sum with the linear term, we identify that the following identity holds by inspection and realizing that the $j = 0$ provides zero contribution to the sum

$$\sum_{k=0}^{(N-1)/2}k^p\binom{N}{k} = \sum_{k=0}^{(N-1)/2-1}\binom{N}{k}(N-k)(k+1)^{p-1}. \tag{12}$$

Writing the expression on the RHS explicitly we obtain

$$\sum_{k=0}^{(N-1)/2} k\binom{N}{k} = N\sum_{k=0}^{(N-1)/2}\binom{N}{k} - \sum_{k=0}^{(N-1)/2} k\binom{N}{k} - \left(\frac{N+1}{2}\right)\binom{N}{\frac{N-1}{2}}, \tag{13}$$

where the last term comes from forcing the sum to run from $j_{\min} = 0$ to $j_{\max} = (N-1)/2$. Now moving the second term on the RHS to the LHS and using Eq. 7 to evaluate the first term we obtain the first non-trivial result written in Eq. 8. The remaining identities follow the same spirit. Plugging Eqs. 7-11 into Eq. 6 and simplifying we obtain

$$\mathcal{M}_2(|W_K^{(0)}\rangle) = 3\log_2(N) - \log_2(7N-6). \tag{14}$$

References

LEONE, Lorenzo; OLIVIERO, Salvatore F.E.; HAMMA, Alioscia. Stabilizer Renyi Entropy. **Physical Review Letters**, American Physical Society, v. 128, n. 5, p. 050402, Feb. 2022.

ODAVIĆ, J. et al. Complexity of frustration: a new source of non-local non-stabilizerness. arXiv, arXiv:2209.10541, Sept. 2022.