Entanglement link representation

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1 Introduction

Starting from the point of view of the area law entanglement present in gapped condensed matter systems, one can test if particular states admit this point of view or if the entanglement has a long-range character. We obtain the entanglement link representation for two different cases. The transverse-field Ising model at the quantum critical point and geometrically frustrated antiferromagnetic state. We observe that both considered cases admit long-range entanglement.

2 Method

We follow the approach from Ref. (ROY et al., 2021). To obtain an optimal link representation of the entanglement of a state we evaluate the following expression

$$AJ = S, \tag{1}$$

which represents a linear system of equations with a J vector containing the unknowns. The unknown link vector J is of size $[\frac{N(N-1)}{2}]$. The operators (matrix) A is of size $[2^{N-1}-1, \frac{N(N-1)}{2}]$. The vector S is of size $[2^{N-1}-1]$. Where N denotes the number of spins or qubits. The first $2^{N-1}-1$ refers to the possible subpartitions a system can be split into and evaluated. The possible splits are also removed because in the case of a pure state the entanglement of a subsystem and its complement are the same. Therefore, we do not need to calculate twice the same value. The removal of the one comes from zero spin subpartition.

It turns out that the Eq (1) system of linear equations is overdetermined, which means that the number of equations is much larger than the number of unknowns. Next, J is the vector of link strengths. So inside we put all the possible pairs of links and order them the way we like. There are $\frac{N(N-1)}{2}$ possible pairs of links and therefore as many unknown values. To obtain something from the linear equation we look at the equation instead

$$A^\dagger AJ = A^\dagger S. \tag{2}$$

If we now solve this system of equations we obtain Js that represent the link strengths. This method scales exponentially in the system size, as the number of partitions we have to consider is $\sim 2^N$.

3 Model

We consider the transverse-field Ising model (TFIM). It is defined by the following Hamiltonian,

$$H = J\sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - h\sum_{j=1}^{N} \sigma_j^z, \tag{3}$$

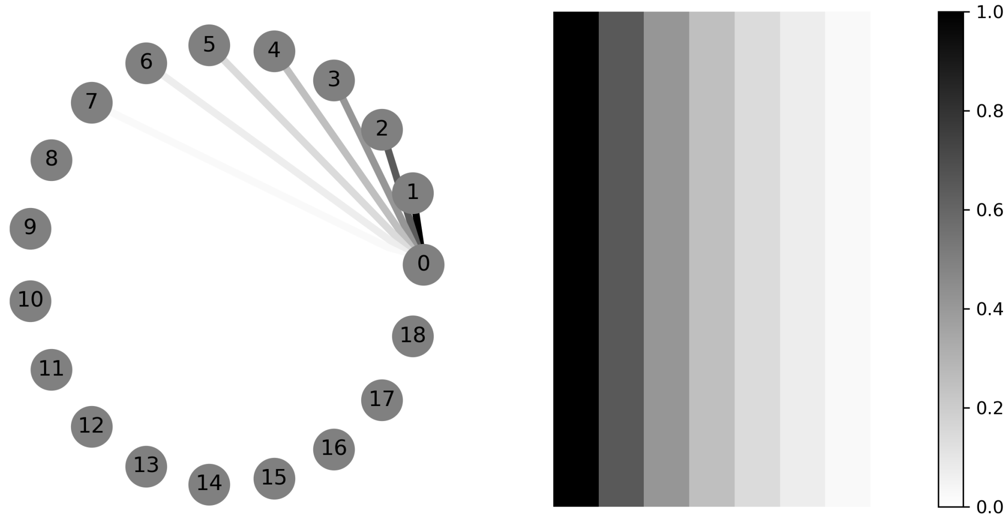

Figure 1: Entanglement link representation the critical model with $N = 19$ spins and the corresponding graph representation on the left. The weights have been normalized to first neighboring link of strength 1. We consider this model critical for $J/h = -1$. On the left the color map coding is presented.

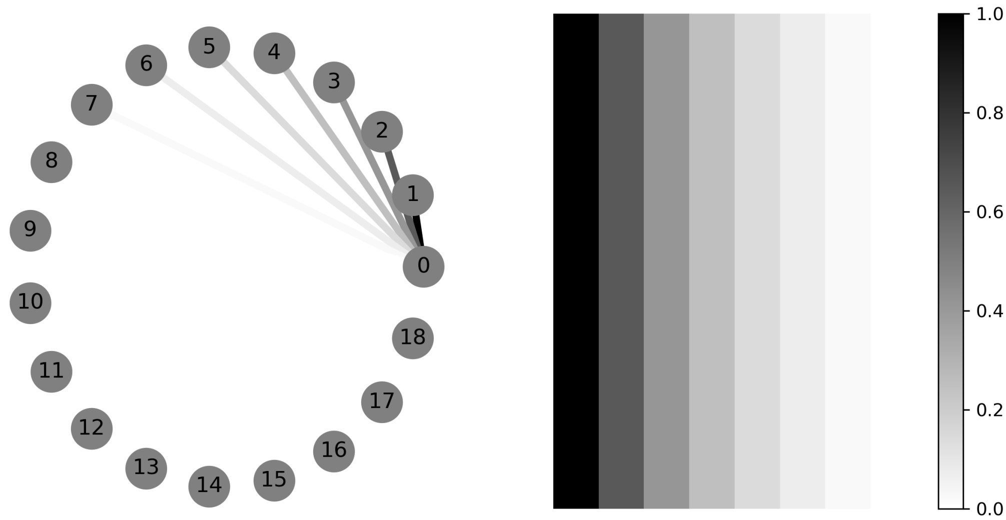

Figure 2: Entanglement link representation the geometrically frustrated model with $N = 19$ spins and the corresponding graph representation on the left. The weights have been normalized to first neighboring link of strength 1. We consider this model frustrated for $J = 100$, $h = 1$. On the left the color map coding is presented.

where the σ^α with $\alpha = x,\ y,\ z$ are the Pauli operators. We assume periodic boundary conditions.

In Fig. 1, 2, and 3 we present the numerical results for the two considered cases. We observe that the slope of the critical cases roughly matches the predicted value of the entanglement

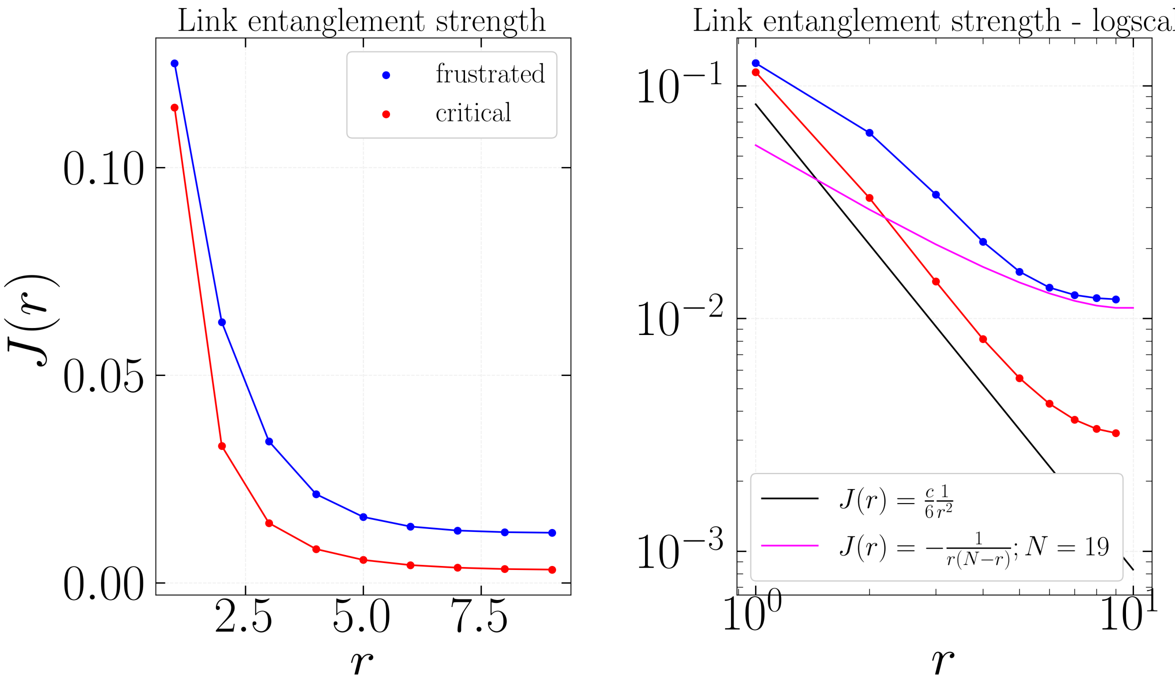

Figure 3: Entanglement link representation for two different cases. For the ground state at the quantum critical point and the geometrically frustrated state with $N = 19$ spins. We consider this model critical for $J/h = -1$ and frustrated for $J = 100$, $h = 1$ an odd number of spins N. See the full phase diagram in Ref. (ODAVIĆ et al., 2022).

link strength (SINGHA ROY et al., 2020) as

$$J(x, y) = \frac{1}{2}\frac{\partial^2 S(x, y)}{\partial x \partial y} \tag{4}$$

where when we plug in the von Neumann entanglement entropy for the critical case $S = \frac{c}{3}\log(x - y)$ we obtain

$$J^{\text{crit}}(x, y) = \frac{c}{6}\frac{1}{(x - y)^2}. \tag{5}$$

Now if apply the same reasoning (FRANCHINI, n.d.) to the frustrated state with the entanglement with the form $S^{\text{frus}} = \log 2 - r\log(r) - (1 - r)\log(1 - r)$ in the classical limit $h \to 0^+$ (GIAMPAOLO; RAMOS; FRANCHINI, 2019), and the suggestion reads

$$S(x, y) = c + \frac{(x - y)}{N}\log\left(\frac{x - y}{N}\right) + \left(1 - \frac{(x - y)}{N}\right)\log\left(1 - \frac{(x - y)}{N}\right), \tag{6}$$

and we obtain after applying Eq. (4) the following guess at the behavior of links in the classical frustrated regime

$$J^{\text{frus}}(x, y) = -\frac{1}{(x - y)(N - (x - y))}. \tag{7}$$

This equation is plotted in Fig. 1 but does not follow the numerical results in behavior. This is probably due to finite-size effects. One way to verify the behavior of the scaling entanglement links in geometrically frustrated chains would be to focus on the kink W state (See IDS 001) and find a way to constructively and exactly evaluate the entanglement of all the partitions. This is a very long calculation, but if executed would yield entanglement link data for much larger system sizes.

References

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